CPSC 421/501 Sept 15

today
- Review strings/words & languages
- State thm: there is no surjection $\Sigma^* \to \mathrm{Power}(\Lambda^*)$ for alphabets Σ, Λ and explain relevance
- Prove this thm using set theory

on Thursday

- Russell's Paradox & Other paradoxes/thms with negation + self-references

today and/or Thursday

All this is in the first handout!

Self-referencing, Uncountability, and Uncomputability

We'll go around 10:20 (50 minutes), breakout rooms in Zoom for some problems (typically from 2019 homework or 2019 webpages
- review midterm
- review final

meet back; discussion/question

I'll post office hours for tomorrow by this afternoon.

Homework teams of up to 4 students;

some problems written up by each student

" " " " by the team

Last time $\left\{ \begin{array}{l} \text{start of handout} \\ \text{Ch. 0 of textbook: Introduction to the} \\ \text{Theory of Computing, by M. Sipser} \\ \text{3rd Edition} \end{array} \right.$

An alphabet, Σ, is any finite, non-empty set.

The elements of Σ are called <u>symbols</u> or <u>letters</u>

textbooks

very common in the literature

Σ^k = k-tuples of elements of Σ

$\Sigma = \{a\}$, $\Sigma^4 = \{(a,a,a,a)\} \overset{\text{more simply}}{=} \{aaaa\}$

or $\{a^4\}$

$$\underbrace{\Sigma \times \Sigma \times \ldots \times \Sigma}_{k^{th} \text{ Cartesian product}}$$

ordered tuples

$\Sigma^0 = \{\varepsilon\}$ $\quad \varepsilon$ = empty string

$\Sigma = \{a,b\}$, $\Sigma^2 = \{aa, ab, ba, bb\}$

literally ab is really (a,b)

ε string!

Concatenation: $aba \circ bb$

$= ababb$

$$\Sigma^k \circ \Sigma^\ell \xrightarrow{\text{concat}} \Sigma^{k+\ell}$$

$\varepsilon \circ aba$

$= aba$

Σ is an alphabet

"set of strings over Σ" notation Σ^* is just

$$\Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \dots = \bigcup_{m=1}^{\infty} \Sigma^m$$

<u>Claim</u>: Any C-program is a string over ASCII alphabet,

$|ASCII| = 256$, has $a, \dots z, A, \dots Z, 0, 1, 2, \dots, 9$

This is mostly true, (it does depend on how you interpret a C program — what built in libraries, etc.)

Similarly, any book in English is an ASCII string (ignore illustrations, etc.)

There are many notions of an algorithm:

(1) C program (2) Javascript program

(3) Turing machine (Ch. 3) (4) Deterministic Finite Automaton DFA (Ch. 1)

All be written — with some understanding — as finite strings over some fixed finite alphabet.

e.g. indentation — you have to agree on some convention

Turing machines & DFAs as well

Compound words: Antidisestablishmentarianism

Algorithms $\xleftrightarrow{\text{interpretation}}$ finite length strings over some alphabet Σ

Language over an alphabet, Σ, is a subset of Σ^*.

e.g.,

$\underline{\Sigma} = \{0, \underline{1, \ldots,} 9\}$, languages over Σ include:

$$\text{PRIMES} = \{2, 3, 5, 7, 11, 13, 17, 19, \ldots\} \subset \Sigma^*$$

literally 17 mean (1,7)

$$\text{SQUARES} = \{1, 4, 9, 16, 25, 36, \ldots\} \subset \Sigma^* = \{0, 1, \ldots, 9\}^*$$

Over $\Sigma = \{a, b\}$

"PALINDROMES OVER $\{a,b\}$"

$$= \{\varepsilon, a, b, aa, bb, aaa, \ldots\}$$

not $\underline{ab}$ or $\underline{ba}$ or acb

$$= \{ \sigma \in \Sigma^* \mid \sigma^{\text{reverse}} = \sigma \}$$

where

$$\sigma = (\sigma_1, \ldots, \sigma_k), \quad \Big| \quad \sigma^{\text{reverse}} = (\sigma_k, \sigma_{k-1}, \ldots, \sigma_1)$$

or $\sigma_1 \ldots \sigma_k$ $\qquad = \sigma_k \sigma_{k-1} \ldots \sigma_1$

A language: e.g. PRIMES

SQUARES

PALINDROME OVER Σ

can be viewed as <u>decision problem</u>

<u>formally</u>: A <u>decision problem over Σ</u> (where Σ is an alphabet)

is any subset of Σ^*. The <u>set</u> of <u>all</u> <u>languages</u>

<u>over</u> Σ = $\text{Power}(\Sigma^*)$

where S any set, $\text{Power}(S)$ = 'set of all subsets of S

Theorem: For any alphabets Σ, Λ,

any map $f : \Sigma^* \longrightarrow \text{Power}(\Lambda^*)$

is not surjective, i.e. there is some $L \in \text{Power}(\Lambda^*)$ s.t. there is no $\sigma \in \Sigma^*$ with $f(\sigma) = L$.

Application: For the notions of "algorithm" in CPSC 421/501 (DFA's, Turing machines, etc.) there are decision problems that can't be solved:

$$f : \Sigma^* \xrightarrow[\text{to solve decision problem}]{\text{interpret as an algorithm}} \begin{cases} \text{invalid descriptions} \\ \text{or} \\ \text{decides some language over } \Lambda \end{cases}$$

Goal is to prove this, and ~~this will~~ it will be useful to speak countable/uncountable sets.

Set: $\{1,2,3\}$, $\{a,b\}$, $\{\text{languages over } \Lambda\}$
$= \text{Power}(\Lambda^*)$

$f : S \to T$ map of sets S, T is

(1) injective if $s_1 \neq s_2$, $s_1, s_2 \in S$ then $f(s_1) \neq f(s_2)$; e.g. student ID numbers —

$\{\text{UBC students}\} \longrightarrow \{0, \ldots, 9\}^8$ $\qquad$ $\left\{\begin{array}{l}\text{"injective"} \\ \text{"into"}\end{array}\right.$ same

(2) surjective if $\forall t \in T$ $\quad \exists s \in S$

for all t in T $\quad$ there is some $s \in S$

s.t. $f(s) = t$.

(3) bijective if it is injective & surjective

(also called one-to-one correspondence)

e.g. $\{1, 2, 3\} \xrightarrow{f} \{$🙂, ☹️, 😖$\}$

$f(1) =$ 😖, $f(2) =$ 🙂, $f(3) =$ ☹️

e.g. <u>surjective</u>

Joel gardens:

Monday ⟶ Mint

Tuesday ⟶ Basil

Wed ⟶ Basil

Th ⟶ Basil

Fr ⟶ Basil

Sat, Sun ⟶ relax

hit everything in T

$f: \{\text{Monday}, \ldots, \text{Sunday}\} \to \{\text{Mint}, \text{Basil}, \text{relax}\}$

$$f: \quad S \longrightarrow T$$

"domain" "range"

f could map

$f: \{\text{Mon}, \ldots, \text{Sun}\} \to \{\text{Mint}, \text{Basil}, \text{relax}, \text{lettuce}\}$

if I never work or lettuce, bad...

Say that S and T have the <u>same</u> <u>size</u> if there is a bijection $f: S \to T$.

We say that T is <u>larger than</u> S if there is no surjection $S \to T$.

S, T finite, and $|S|$ = # of elements in S

$\exists$ bijection $S \to T$ iff $|S| = |T|$

but

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

$\mathbb{Q} = \mathbb{Q}_{>0} = \{\text{positive rationals}\}$

$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$

} all have same size

e.g. even thought $\mathbb{N} \underset{\text{proper}}{\subset} \mathbb{Z}$

$1, 2, 3, 4, 5, \ldots$ ← $\mathbb{N}$ naturals

bijection

$0, 1, -1, 2, -2, \ldots$ ← $\mathbb{Z}$ integers

Questions for discussion (random breakout)

1) HW:

- do the followings sets have the same size as $\mathbb{N}$

Hint! Has the same size — $\{\text{functions from } \{0,1\} \to \mathbb{N}\}$

this one ~~doesn't~~ $\{$ functions from $\mathbb{N} \to \{0,1\}$ $\}$

(2) Give an "intuitive example" of injection / surjection

(3) Come up with a surprising injection / surjection / bijection

(4) If f, g are surjections, $f: S \to T$
$g: T \to U$

is $g \circ f: S \to U$ and why? can you prove?

You can currently do (2), (3), (4) ←

Next time we will do (1);