Individual Homework

Problem (1)

Suppose TM $H$ decides $\text{HALT}_{TM}$, that is given input $<M, w>$ (where $M$ is a description of a TM and $w$ is the input to $M$), $H$ will accept $<M, w>$ if and only if $M$ halts on input $w$. Otherwise $H$ will reject $<M, w>$.

Define a new TM $D$ with such a procedure:
1 - If the input is not of form $<M>$ (a TM), then reject
2 - Otherwise, Simulate $<M,<M>>$ on $H$
3 - If $H$ accepts $<M,<M>>$, then loop forever
4 - If $H$ rejects $<M,<M>>$, then accept

Now consider TM $D$ with input $<D>$. We have three cases:

Case 1 - $D$ rejects $<D>$: Then $<D>$ is not a description of a TM, which is a contradiction.

Case 2 - $D$ accepts $<D>$: Then $H$ must reject $<D,<D>>$, which means $D$ does not halt on input $<D>$. Which is a contradiction.

Case 3 - $D$ loops on input $<D>$: Then $H$ must accept $<D,<D>>$, which means $D$ halts on input $<D>$. Which is a contradiction.

In all cases we got a contradiction, therefore our initial assumption was incorrect and $\text{HALT}_{TM}$ is undecidable.
Problem (2)

HALT$_M$ is recognizable, for any TM $M$ and input $w$, simulate running $M$ on $w$ using a universal Turing machine, and accept if $M$ halts.

The complement of HALT$_M$ is not recognizable. Suppose it was recognizable, since we know HALT$_M$ is recognizable, then HALT$_M$ would be decidable which is a contradiction.
Group Homework

Problem (1)

(a)

Suppose $L$ is decided by a TM $H$. That is given a valid description of a TM, $<M>$, $H$ will accept $<M>$ iff $M$ accepts some input, otherwise $H$ will reject.

Consider TM $D$ which will decide $A_{TM}$. $D$ has inputs of form $<<M>, w>$ (a TM and its input):

1 - If the input is not of form $<<M>, w>$, then reject
2 - Otherwise, Construct new TM $M'_w$ as such:
   2.1 - If the input to $M'_w$ is not $w$, then $M'_w$ rejects
   2.2 - Otherwise $M'_w$ will simulate running $M$ with input $w$.
   ($M'_w$ will accept, reject, or loop on input $w$ if $M$ does the same on $w$)
3 - Simulate running $H$ with input $<M'_w>$.
4 - If $H$ accepts, accept. Otherwise, reject.
   (Since $H$ is a decider it will not loop)

In the description of $D$, $M'_w$ is such that the only input it will possibly accept is $w$. $M'_w$ will accept $w$ iff $M$ accepts $w$. Since $H$ is a decider, if $H$ accepts $<M'_w>$, then $M$ accepts $w$. Otherwise $M$ rejects $w$. This means TM $D$ decides $A_{TM}$. Which is a contradiction, therefore our initial assumption was incorrect and $L$ is undecidable.
(b)

$L$ is recognizable. For any TM $M$, the set of inputs of size $i$ are finite since the alphabet of $M$ is finite. Initialize $i$ to 1 and simulate running $M$ with a universal TM on all inputs of size less than $i$ for $i$ steps. If any input is accepted by $M$ in $i$ steps (or less) accept $M$, otherwise increment $i$ and try again. Assuming that $L$ is not empty, this method will halt and accept $M$ iff $M \in L$.

(c)

No, the complement of $L$ is not recognizable. Similar to problem 2 of the individual section of homework 7, if the complement of $L$ were to be recognizable, then $L$ would be decidable, since $L$ is recognizable. This is a contradiction.
Yes, NP is closed under concatenation. Suppose you have two languages $L_1$ and $L_2$ in NP, then there are non-deterministic TMs $M_1$ and $M_2$ that decide $L_1$ and $L_2$ in polynomial number of steps in the size of their inputs. Let $L_3$ be \{ $l_1 \circ l_2$ | $l_1 \in L_2, l_2 \in L_2$ \}. Then we can describe a non-deterministic TM $M_3$ that decides $L_3$, it cuts the input at all possible locations and checks if the parts are members of $L_1$ and $L_2$. More concretely the $M_3$ works as such:

1 - For all possible cuts to the input $w$ into parts $w_1$ and $w_2$ such that $w_1 \circ w_2 = w$ do the following:
   2 - Simulate running $M_1$ with input $w_1$
   3 - If $M_1$ rejects $w_1$, then reject
   4 - Otherwise, simulate running $M_2$ with input $w_2$
      5 - If $M_2$ accepts $w_2$, then accept. Otherwise reject.

$M_3$ only reaches the accepts $w$ state iff there exist $w_1$ and $w_2$ such that $w = w_1 \circ w_2$ and $w_1 \in L_1$ and $w_2 \in L_2$, so $M_3$ decides $L_3$. There only $|w|$ of these cuts possible and since $M_1$ and $M_2$ take NP steps, $M_3$ takes at most a factor of $w$ more steps, which means $L_3$ is in NP.
Problem (3)

Yes, NP is closed under intersection. Suppose you have two languages $L_1$ and $L_2$ in NP decided by non-deterministic TMs $M_1$ and $M_2$. Using a universal non-deterministic TM simulate $M_1$ on input $w$. If $M_1$ rejects, reject. Otherwise simulate $M_2$ on input $w$. If $M_2$ rejects, reject. Otherwise accept. The procedure takes NP steps and decides $L_1 \cap L_2$.

Problem (4)

P is closed under complement. For an language $L$ in P, let M be the deterministic TM that decides it in polynomial time. We construct a TM $M'$ that decides the complement of $L$ in polynomial time:

$M'$ on input $w$:
1 - Simulate running $M$ with input $w$
    2 - If $M$ accepts $w$, then reject
    3 - If $M$ rejects $w$, then accept

Above we proved that P is closed under complement, since we’ve assumed that P = NP, then NP is also closed under complement. From class we know SAT is in NP, then the complement of SAT is also in NP.