INDIVIDUAL HOMEWORK 1, CPSC 421/501, FALL 2020

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) Homework that is difficult to read may not be graded.
(4) You may work together on homework in groups of up to four, **but you must write up your own solutions individually and must acknowledge with whom you worked.** You must also acknowledge any sources you have used beyond the textbook and two articles on the class website.

At the end of this document there are a few sample problem with solution.

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(1) Which of the following maps are injections (i.e., one-to-one), which are surjections (i.e., onto), and which are both (i.e., bijections or one-to-one correspondences)? Justify your answer.
   (a) $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = x + 9$.
   (b) $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x + 9$.

(2) If $f\colon S \to T$ and $g\colon T \to U$ are both injective (i.e., one-to-one), is $g \circ f$ (which is a map $S \to U$) necessarily injective? Justify your answer.

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Bonus Question, Worth an Extra 10 out of 100 Points for Homework 1

[Solutions to this problem will not be released. Bonus questions tend to be more difficult than the usual course material and will not appear on any exam.]

(3) (Cancellation Property)

Left Cancellation: We say that a map (of sets) $f\colon S \to T$ has the *left cancellation property* if for any two maps g, h from a set $U \to S$ we have $fg = fh$ (i.e., the map $f \circ g\colon U \to T$ equals the map $f \circ h$) implies that $g = h$. Show that this property holds of f iff f is injective.

Right Cancellation: Formulate a similar *right cancellation property* for a map $f\colon S \to T$ and show that it is equivalent to f being surjective.

Sample Exercises With Solutions:

People often ask me how much detail they need in giving explanations for the homework exercises. Here are some examples. The material in brackets [like this] is optional.

Sample Question Needing a Proof: If $f\colon S \to T$ and $g\colon T \to U$ are surjective (i.e., onto) is $g \circ f$ (a map $S \to U$) is necessarily surjective? Justify your answer.

Answer: Yes.

[To show that $g \circ f$ is surjective, we must show that if $u \in U$, then there is an $s \in S$ such that $(g \circ f)(s) = u$.]

If $u \in U$, then since g is surjective there is a $t \in T$ such that $g(t) = u$. Since f is surjective, there is an $s \in S$ such that $f(s) = t$. Hence

$$(g \circ f)(s) = g(f(s)) = g(t) = u.$$

Therefore each $u \in U$ is $g \circ f$ applied to some element of S, and so $g \circ f$ is surjective.

Sample Question Needing a Counterexample: If $f\colon S \to T$ is injective, and $g\colon T \to U$ is surjective, is $g \circ f$ is necessarily injective? Justify your answer.

Answer: No.

[To show that $g \circ f$ is not necessarily injective, we must find one example of such an f and g where $g \circ f$ is not injective.]

Let $S = T = \{a, b\}$ and $U = \{c\}$; let $f\colon S \to T$ be the identity map (i.e., $f(a) = a$ and $f(b) = b$), and let $g\colon T \to U$ (there is only one possible g in this case) be given by $g(a) = g(b) = c$.

Then f is injective (since $f(a) \neq f(b)$) and g is surjective, since $U = \{c\}$ and $c = g(a)$). However $g \circ f$ is not injective, since $(g \circ f)(a) = c = (g \circ f)(b)$.

Injectivitiy and Surjectivity of a Given Map: If $f\colon \mathbb{N} \to \mathbb{N}$ is given by $f(n) = 2n + 5$, is f injective? Is f surjective?

Answer: f is injective, because if $f(n_1) = f(n_2)$, then $2n_1 + 5 = 2n_2 + 5$ and therefore $n_1 = n_2$.

[Hence f maps distinct values of $\mathbb{N}$ to distinct values of $\mathbb{N}$, i.e., $n_1 \neq n_2$ implies that $f(n_1) \neq f(n_2)$.]

f is not surjective, because there is no value $n \in \mathbb{N}$ such that $f(n) = 1$: if such an n existed, then $2n + 5 = 1$ and so $n = -2$ which is not an element of $\mathbb{N}$.

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