CPSC 421/501: Start Ch 1 (regular languages) of [Sip] on Wednesday

- 2 or 3 more paradoxes/theorems

- Countably infinite and uncountable sets

There is no surjection $\mathbb{N} \to \text{Power}(\mathbb{N})$

$\{1,2,3\} \to \text{Power}(\{1,2,3\})$

$A^* \to \text{Power}(A^*)$

If there is a surjection $S \to T$ then "$|S| \geq |T|$"

If there is an injection $S \to T$ then "$|S| \leq |T|$"

(5) Russell's Paradox: "Let S be the set of all sets that do not contain themselves." Serious problem; a lot of people worked to fix this. Problem: Does S contain itself???

(1) If $S \in S$ then S is a set that does not contain itself, i.e. $S \notin S$

(2) If $S \notin S$ then $S \in S$.

Can a set really ever contain itself, can $\{1,2,3\} \in \{1,2,3\}$??

$\{\{1,2,3\}\}$??

Usual idea: probably all sets, S, don't contain themselves.

Russell's Paradox: "Let S be the set of all sets that do not contain themselves." Serious problem; a lot of people worked to fix this.

Should be all sets.

Remedy: "All sets" is a class (something that be bigger than a set)

Needed to be solved

Remark: We think of injections, bijections

bijection: $f: \underbrace{\{a, b, c\}}_{S} \longrightarrow \underbrace{\{\alpha, \beta, \gamma\}}_{T}$

$a \mapsto \alpha$

$b \mapsto \beta$ $\rightsquigarrow$ bijection $S^* \longleftrightarrow T^*$

$c \mapsto \gamma$ bijection $\text{Power}(S^*) \longleftrightarrow \text{Power}(T^*)$

However, all we really need are <u>surjections</u>.

Gödel Incompleteness: Say "you can build" a sentence S that "means" "there is no proof that S is true" $= S$

① If S is true: there is no proof that S is true. "incomplete"

② " " " false: there is a proof that S is true "inconsistent"

assume this doesn't happen

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⑦ Halting problem: self reference & negation, but it turns into a theorem.

<u>Countably infinite</u> & <u>Uncountable</u>:

{Algorithms} (all the ones in [Sip]) is countable

{Decision problem} (" " " " ") is uncountable

Tools:

① If S is countable, and there is a bijection $S \to T$, then T is countable.

Proof: S is countable, hence there is a surjection $\mathbb{N} \to S$. Then $\mathbb{N} \to S \longrightarrow T$ gives surjection $\mathbb{N} \to T$.

①' What if we have a surjection $S \longrightarrow T$ and S is countable? Then

$$\mathbb{N} \xrightarrow{\text{surj}} S \xrightarrow{\text{surj}} T \quad \text{gives} \quad \mathbb{N} \xrightarrow[\text{a surjection}]{\text{must be}} T$$

Convenient to notice: the compositions of 2 surjections is a surjection. (Worked out)

Homework 1: The composition of 2 injections is an injection

If A is an alphabet, then A^* is countably infinite.

Proof 1: E.g. $A = \{a, b, c\}$,

$$A^* = \{ \varepsilon, \quad a, b, c \quad , \quad aa, ab, ac, \dots, cc, \quad aaa, \dots \}$$

$$\begin{array}{ccccccccc} & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\ \mathbb{N} & 1 & 2 & 3 & 4 & 5 & 6 & \dots \end{array}$$

Proof 2: Associate to aabacaccb

$a \leftrightarrow 1$
$b \leftrightarrow 2$
$c \leftrightarrow 3$

$$\rightsquigarrow 1\ 1\ 2\ 1\ 1\ 1\ 3\ 3\ 2$$

$$\rightsquigarrow 2^1\, 3^1\, 5^2\, 7^1\, 11^1\, 13^1\, 17^3\, 19^3\, 23^2$$

Gives $\{1,2,3\}^* \xrightarrow[\text{injective}]{} \mathbb{N}$ should exist surjection

$\mathbb{N} \to \{1,2,3\}^*$

Since $A^* \overset{\text{bijection}}{\longleftrightarrow} \mathbb{N}$ then

$\text{Power}(A^*) \overset{\text{bijection}}{\longleftrightarrow} \text{Power}(\mathbb{N}) \leftarrow$ uncountable

=

Power($\mathbb{N}$) = PRIMES, EVEN, ODD, DIVISIBLE_BY_2

, " " 3, ...

$\{1,2,3\} \to \text{Power}(\{1,2,3\})$: $1 \in f(1)$, $2 \notin f(2)$, $3 \in f(3)$