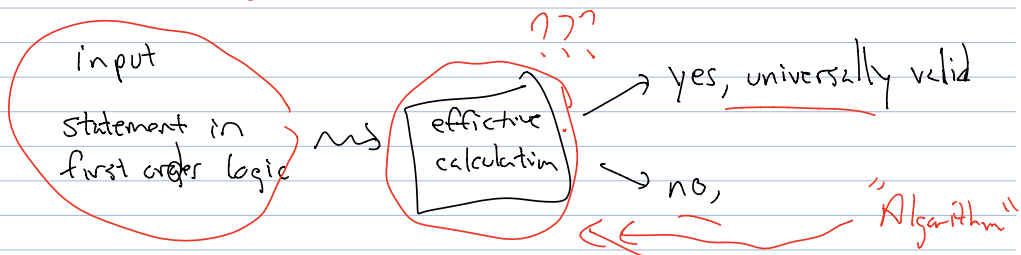


≈ 1900 Hilbert's 23 problems (Thanks via T.)

≈ 1928 Hilbert & Ackermann: Entscheidungsproblem: Is there an "effective calculation"



≈ 1936, solutions by Church & Turing

- HW #1
- 8 Exercises
  - Beyond Exercises
  - Sample Exercises with solutions
- ← gradescope.com

Today & Monday

- Cantor's Theorem
- Countability, Uncountability
- Self-references + negation in paradoxes & theorems
- (Sip) Diagonalization

Cantor's Thm "☹️" If  $S$  is a set, there is no surjection  $f: S \rightarrow \text{Power}(S)$  = set of all subsets of  $S$ .

Cantor's Thm ☺️ ☹️: If  $S$  is a set, and

$f: S \rightarrow \text{Power}(S)$ , then

$$T = \{s \in S \mid s \notin f(s)\}$$

is not in the image of  $f$ , i.e.  $T \neq f(s)$  for all  $s \in S$

Example:  $S = \{1, 2, 3, 4\}$  and  $f$  has

$$1 \in f(1) \quad 2 \in f(2) \quad 3 \in f(3) \quad 4 \in f(4)$$

e.g.  $f(1) = \{1, 2\}$      $f(2) = \{2, 3\}$      $f(3) = \{1, 2, 3, 4\}$      $f(4) = \{4\}$

$|S| = 4$      $S = \{1, 2, 3, 4\}$   
 $|\text{Power}(S)| = 16$      $\text{Power}(S) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \dots, \{2, 3, 4\}, \{1, 2, 3, 4\} \}$

Claim:  $f(1) \neq \emptyset$  just because  $1 \in f(1)$   
 $f(2) \neq \emptyset$  " "  $2 \in f(2)$   
 $f(3) \neq \emptyset$  " " "  
 $f(4) \neq \emptyset$  " " "

$$T = \{s \in S \mid s \notin f(s)\} = \emptyset$$

e.g. Say  $1 \in f(1), 2 \notin f(2), 3 \notin f(3), 4 \in f(4)$

$$T = \{ \quad \quad \quad 2 \quad , \quad 3 \quad \quad \quad \}$$

$1 \in f(1) \quad 1 \notin T$  so  $f(1) \neq T$

$2 \notin f(2) \quad 2 \in T$  so  $f(2) \neq T$

$\vdots$

Proof: Say that  $f(t) = T$  for some  $t$ . Then  $\rightarrow$  paradox

Either  $t \in f(t) = T$  ... paradox (contradiction)

$t \notin f(t)$  ... " "

$$t \in f(t) = T \Rightarrow t \notin f(t)$$

$$t \notin f(t) = T = \{ s \mid s \notin f(s) \}$$

$$t \notin \{ s \mid s \notin f(s) \} \Rightarrow$$

$$\boxed{t \in \{ s \mid s \in f(s) \}} \Rightarrow t \in f(t)$$

Corollary: There is no surjection  $\mathbb{N} \rightarrow \text{Power}(\mathbb{N})$

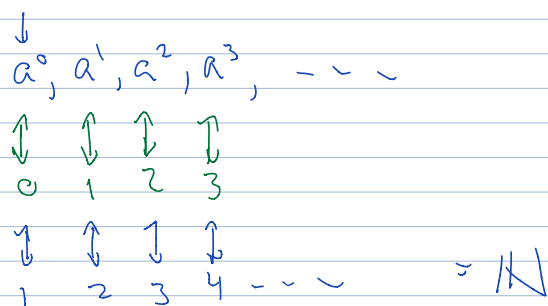
" " " " "  $A^* \rightarrow \text{Power}(A^*)$

= Languages over  $A$

Uncountable:  $\text{Power}(\mathbb{N})$ ; also  $\text{Power}(A^*)$

Claim:  $A$  alphabet,  $A^*$  is countably infinite.

Say  $A = \{a\}$ .  $A^* = \{\epsilon, a, aa, aaa, aaaa, \dots\}$



$\mathbb{N} \rightarrow \{a\}^*$

$n \mapsto a^{n-1} = \underbrace{a \dots a}_{n-1 \text{ times}}$

base<sub>3</sub> numbers  $0, 1, 2, \cancel{10}, 11, 12, \cancel{20}, 21, 22, 100, \dots$

$\{a, b\}^* = \{\epsilon, a, b, aa, ab, ba, bb, \dots\}$

Gödel:  $\{a, b, c, d\}$

bae bca daa

2 1 1 2 1 1 4 1 1

