≈ 1900 Hilbert's 23 problems (Thanks via T.)

≈ 1928 Hilbert & Ackermann: Entscheidungsproblem: Is there an "effictive calculation"

input

statement in first order logic ⇝ effective calculation ??? → yes, universally valid

→ no,

"Algorithm"

"Turing machines"

≈ 1936, solutions by Church & Turing

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HW #1
- 8 Exercises ← gradescope.com
- Beyond Exercises
- Sample Exercises with solutions

Today & Monday
- Cantor's Theorem
- Countability, Uncountability
- Self-references + negation in paradoxes & theorems
- (Sip) Diagonalization

Cantor's Thm " " If S is a set, there is no surjection $f: S \to \mathrm{Power}(S)$ = set of all subsets of S.

Cantor's Thm : If S is a set, and $f: S \to \mathrm{Power}(S)$, then

$$T = \{ s \in S \mid s \notin f(s) \}$$

is not in the image of f, i.e. $T \neq f(s)$ for all $s \in S$

Example: $S = \{1,2,3,4\}$ and f has

$$\boxed{1 \in f(1) \qquad 2 \in f(2) \qquad 3 \in f(3) \qquad 4 \in f(4)}$$

e.g. $f(1) = \{1,2\}$ $f(2) = \{2,3\}$ $f(3) = \{1,2,3,4\}$ $f(4) = \{4\}$

$|S| = 4$

$|\mathrm{Power}(S)| = 16$

$S = \{1,2,3,4\}$

$\mathrm{Power}(S) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \ldots, \{2,3,4\}, \{1,2,3,4\} \}$

Claim: $f(1) \neq \emptyset$ just because $1 \in f(1)$

$f(2) \neq \emptyset$ " " $2 \in f(2)$

$f(3) \neq \emptyset$ " " "

$f(4) \neq \emptyset$ " " "

$$T = \{ s \in S \mid s \notin f(s) \} = \emptyset$$

e.g. Say $1 \in f(1)$, $2 \notin f(2)$, $3 \notin f(3)$, $4 \in f(4)$

$$T = \{ \quad 2 \;,\; 3 \quad \}$$

$1 \in f(1)$ $\quad 1 \notin T$ so $f(1) \neq T$

$2 \notin f(2)$ $\quad 2 \in T$ so $f(2) \neq T$

$\vdots$

Proof: Say that $f(t) = T$ for some t. Then $\leadsto$ paradox

Either $\quad t \in f(t) = T$ ... paradox (contradiction)

$\qquad\quad t \notin f(t)$ ... " $\quad$ "

$t \in f(t) = T \Rightarrow t \notin f(t)$

$t \notin f(t) = T = \{ s \mid s \notin f(s) \}$

$t \notin \{ s \mid s \notin f(s) \} \Rightarrow$

$t \in \{ s \mid s \in f(s) \} \Rightarrow t \in f(t)$

Corollary: There is no surjection $\mathbb{N} \to \text{Power}(\mathbb{N})$

" $\quad$ " $\quad$ " $\quad$ " $\quad$ " $\quad A^* \to \text{Power}(A^*)$

$= \text{Languages over } A$

Uncountable: Power($\mathbb{N}$); also Power(A^*)

Claim: A alphabet, A^* is countably infinite.

Say $A = \{a\}$. $\quad A^* = \{\varepsilon, a, aa, aaa, aaaa, \dots\}$

$$\begin{array}{ccccc} a^0, & a^1, & a^2, & a^3, & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 1 & 2 & 3 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & \dots \end{array} = \mathbb{N}$$

$$\mathbb{N} \longrightarrow \{a\}^*$$

$$n \longmapsto a^{n-1} = \underbrace{aa\dots a}_{n-1 \text{ times}}$$

base_3 numbers $\quad 0, 1, 2, \cancel{10}, 11, 12, \cancel{20}, 21, 22, 100, \dots$

$$\{a,b\}^* = \{\varepsilon, a, b, \quad aa \quad ab \quad ba \quad bb\}$$

Gödel: $\quad \{a, b, c, d\}$

baabaadaa

2 1 1 2 1 1 4 1 1

$$2^2\, 3^1\, 5^1\, 7^2\, 11^1\, 13^1\, 17^4\, 19^1\, 23^1$$