

§4.2 [S:1]

(Halting problem
undecidable)

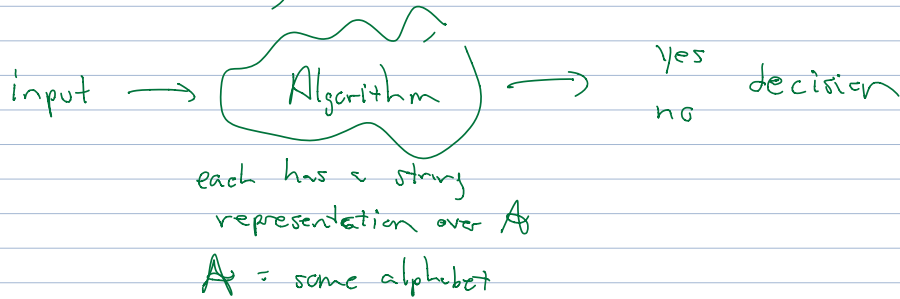
Section 1
self-referencing + recursion

(IR uncountable
Some problems have no solution)

Section 2

(The connection)

Where are we going



A^*
strings

some are
algorithms

$S \subset A^*$

$f: S \rightarrow \{\text{decision problems}\}$

$s \mapsto$ the decision problem
it solves

Countability: $f: S \rightarrow \{\text{decision problems}\}$

Claim! f is not a surjection. Proof: S countable

but $\{\text{decision problems}\}$
uncountable

S countable and $\{\text{decision problems}\}$ uncountable,
then f cannot be a surjection.

Definition If S is a set, then the power set of S , denoted $\text{Power}(S)$ (or " 2^S ") is the set of all subsets of S .

$$\text{Power}(\{a, b, c\}) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}$$

$$8 = 1 + 3 + 3 + 1$$

If S is finite $|\text{Power}(S)| = 2^{|S|} = \text{"2 to the power } |S|\text{"}$

=

$$\{\text{decision problems}\} = \{\text{all subsets of } A^*\}$$

↑
fix some finite alphabet A

$$A^* = \text{all strings, } \boxed{\text{decision problem: } A^* \rightarrow \{\text{no, yes}\}}$$

we view $A^* \rightarrow \{\text{no, yes}\}$ ↪ the subset of A^* where you say "yes"

e.g. $\text{PRIMES} = \{ s \in \{0, 1, \dots, 9\}^* \mid \overset{\text{yes}}{s} \text{ represents a prime} \}$

$$\text{So } \{\text{decision problems}\} \leftrightarrow \{\text{subset of } A^*\} = \text{Power}(A^*)$$

↑
look where the answer is "yes"

$$A^* = A^0 \cup A^1 \cup A^2 \cup \dots \quad A^k = \text{strings of length } k \text{ over } A$$

$$\{a, b\}^* = \{ \epsilon, a, b, aa, ab, ba, bb, \dots \}$$

Claim: $\text{Power}(A^*)$ is uncountable.

$$\{a\}^* = \{ \epsilon, a, aa, aaa = a^3, aaaa = a^4, \dots \}$$

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Cantor's Thm: If $f: S \rightarrow \text{Power}(S)$, then

f is not surjective.

=

$$\text{e.g. } S = \{a, b, c\} \quad |S| = 3 \quad |\text{Power}(S)| = 2^3 = 8$$

for S finite, $|S| = n \rightsquigarrow n < 2^n$.

=

Pf of Cantor's Thm: Let $f: S \rightarrow \text{Power}(S)$.

$$\text{Let } T = \left\{ s \in S \mid s \notin \underbrace{f(s)}_{\text{subset of } S} \right\}.$$

Imagine that t s.t. $f(t) = T$ (if no such t exists then f is not a surjection).

Either $t \in T \Rightarrow t \notin f(t) = T$ impossible

or $t \notin T \Rightarrow t$ does not satisfy $t \notin f(t) \Rightarrow t \in f(t) = T$ impossible.

Example $S = \{a, b, c\}$. Pick $f(a) = \{a, b\}$

$f(b) = \{a, b, c\}$

$f(c) = \{b\}$

$$T = \{s \in S \mid s \notin f(s)\} = \{c\}$$

$a \in f(a) = \{a, b\}$ yes

$b \in f(b) = \{a, b, c\}$ yes

$c \in f(c) = \{b\}$ no

↑
don't put
a in

contains	$f(a)$	$f(b)$	$f(c)$
a	yes		
b		yes	
c			no

don't put a in T

don't put b in T

do put c in T