§4.2 [Sip]

(Halting problem undecidable } { Section 1 self-referencing + negation

($\mathbb{R}$ uncountable
Some problems have no solution } Section 2

(The connection

Where are we going

input ⟶ Algorithm ⟶ yes / no decision

each has a string representation over A

A = some alphabet

A^* strings some are algorithms $S \subset A^*$

$$f : S \longrightarrow \{\text{decision problems}\}$$

$$s \longmapsto \text{the decision problem it solves}$$

Countability : $f : S \longrightarrow \{\text{decision problems}\}$

Claim: f is not a surjection. Proof: S countable but $\{\text{decision problems}\}$ uncountable

S countable and $\{\text{decision problems}\}$ uncountable, then f cannot be a surjection.

Definition If S is a set, then the power set of S, denoted $\text{Power}(S)$ (or "2^S") is the set of all subsets of S.

$$\text{Power}(\{a,b,c\}) = \{\emptyset, \{a\},\{b\},\{c\}, \{a,b\},\{a,c\},\{b,c\}, \{a,b,c\}\}$$

$$8 = 1 + 3 + 3 + 1$$

If S is finite $|\text{Power}(S)| = 2^{|S|}$ = "2 to the power $|S|$"

$=$

$\{\text{decision problems}\} = \{\text{all subsets of } A^*\}$

fix some finite alphabet A

A^* = all strings, $\boxed{\text{decision problem}: A^* \longrightarrow \{\text{no, yes}\}}$

we view $A^* \longrightarrow \{\text{no, yes}\}$ $\longleftrightarrow$ the subset of A^* where you say "yes"

e.g. $\text{PRIMES} = \{ s \in \{0,1,\ldots,9\}^* \mid \text{yes, } s \text{ represents a prime}\}$

So $\{\text{decision problems}\} \longleftrightarrow \{\text{subset of } A^*\}$
$= \text{Power}(A^*)$

look where ~~the~~ answer is "yes"

$A^* = A^0 \cup A^1 \cup A^2 \cup \ldots$ A^k = strings of length k over A

$\{a,b\}^* = \{ \varepsilon, a, b, aa, ab, ba, bb, \ldots \}$

Claim: $\text{Power}(A^*)$ is uncountable.

$\{a\}^* = \{ \varepsilon, a, aa, aaa = a^3, aaaa = a^4, \ldots \}$

Cantor's Thm: If $f: S \to \text{Power}(S)$, then f is not surjective.

e.g. $S = \{a,b,c\}$ $\quad |S| = 3 \quad |\text{Power}(S)| = 2^3 = 8$

for S finite, $|S| = n \rightsquigarrow n < 2^n$.

Pf of Cantor's Thm: Let $f: S \to \text{Power}(S)$.

Let $T = \{ s \in S \mid s \notin \underbrace{f(s)}_{\text{subset of } S} \}$.

Imagine that t s.t. $f(t) = T$ (if no such t exists then f is not a surjection.

Either $t \in T \Rightarrow t \notin f(t) = T$ impossible

or $t \notin T \Rightarrow t$ does not satisfy $t \notin f(t) \Rightarrow t \in f(t) = T$ impossible.

Example $S = \{a, b, c\}$. Pick $f(a) = \{a, b\}$

$f(b) = \{a, b, c\}$

$f(c) = \{b\}$

$$T = \{ s \in S \mid s \notin f(s) \} = \{ \quad c \quad \}$$

don't put a in

$a \in f(a) = \{a, b\}$ yes

$b \in f(b) = \{a, b, c\}$ yes

$c \in f(c) = \{b\}$ no

containment	$f(a)$	$f(b)$	$f(c)$
a	yes		
b		yes	
c			no

don't put a in T

don't put b in T

do put c in T