HOMEWORK 2, CPSC 421/501, FALL 2019

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) Homework that is difficult to read may not be graded.
(4) You may work together on homework, **but you must write up your own solutions individually.** You must acknowledge with whom you worked. You must also acknowledge any sources you have used beyond the textbook and two articles on the class website.

In these exercises, "the handout" refers to the article "Self-referencing, Uncountability, and Uncomputability" on the 421/501 homepage.

(1) Let $f\colon \mathbb{N} \to \mathrm{Power}(\mathbb{N})$ be given by

$$f(n) = \bigl\{m \in \mathbb{N} \mid m^2 > 3n+4\bigr\}.$$

For example

$$f(1) = \bigl\{m \mid m^2 > 7\bigr\} = \{3,4,5,\ldots\}$$
$$f(10) = \bigl\{m \mid m^2 > 34\bigr\} = \{6,7,8,\ldots\}.$$

(a) For which n is $n \in f(n)$? (From the examples above we see that $1 \notin f(1)$ and $10 \in f(10)$.)
(b) Based on your answer to the previous question, describe the set

$$T = \{s \in \mathbb{N} \mid s \notin f(s)\}.$$

(c) Explain why $1 \notin f(1)$ implies that the above set T cannot equal $f(1)$.
(d) Explain why $10 \in f(10)$ implies that the above set T cannot equal $f(10)$.

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(2) Our first definition of a *countable set* was set, S, such that there is a surjection $\mathbb{N} \to S$. Using this definition, answer the following questions and **justify your answer**. (You may use the fact that the composition of two surjections is a surjection.)
  (a) Show that if there is a bijection $f\colon T \to U$, then T is countable iff (if and only if) U is countable.
  (b) Say that T is countable and there is a surjection $f\colon T \to U$. Show that U is necessarily countable.
  (c) Say that U is countable and there is a surjection $f\colon T \to U$. Show (by giving an example) that T is not necessarily countable.
  (d) Say that U is *uncountable* and there is a surjection $f\colon T \to U$. Show that T is necessarily uncountable.

(3) Our first definition of a *countable set* was set, S, such that there is a surjection $\mathbb{N} \to S$. Using this definition, prove the following statements.
  (a) If S and T are both countable sets, show that $S \cup T$ is countable. (In other words, given a sujection $f\colon \mathbb{N} \to S$ and $g\colon \mathbb{N} \to T$, describe a surjection $\mathbb{N} \to S \cup T$.)
  (b) Show that if W is an uncountable set, and U is a countable set with $U \subset W$ (U is a subset of W), then

$$W \setminus U \stackrel{\text{def}}{=} \{w \in W \mid w \notin U\}$$

is uncountable.

(4) Which of the following sets are countable, and which are uncountable? Justify your answer. [You are allowed to use the results of Problem 2.]
  (a) Power(A^*) where A is an alphabet.
  (b) Power($\mathbb{Z}$).
  (c) $\mathbb{Z}^2$, i.e., $\mathbb{Z} \times \mathbb{Z}$.
  (d) $\mathbb{Z} \times \mathbb{R}$.
  (e) The set of maps $\{0,1,2\} \to \mathbb{Z}$.
  (f) The set of maps $A^* \to \{0,1\}$ where A is an alphabet.
  (g) The set of maps $A^* \to \{0,1,2\}$ where A is an alphabet.

(5) Give a DFA's for the following languages and explain how they work.
  (a)

$$\bigl\{s \in \{a,b\}^* \mid \text{the number of } a\text{'s in } s \text{ is odd}\bigr\}$$

  (b)

$$\bigl\{s \in \{a,b\}^* \mid s \text{ contains } baa \text{ as a substring}\bigr\}$$

  (c)

$$\text{DIV-BY-4} = \bigl\{s \in \{0,1,\ldots,9\}^* \mid s \text{ represents an integer in base 10 divisible by 4}\bigr\}$$

with the conventions used in class for DIV-BY-2 and DIV-BY-3: the empty string is not in DIV-BY-4, the string 0 is in DIV-BY-4, and any other string that begins in 0 is not in DIV-BY-4 (i.e., leading 0's are not allowed). (Also, the rightmost digit is the least significant digit, i.e., the 1's digit, and the leftmost digit is the most significant digit.)

(End of Homework Problems to be Submitted for Credit.)

Exercises Beyond the Homework (not for credit, solutions will not be released):

Linear Algebra and Regular Languages: For a language, L, over an alphabet Σ, let $\mathrm{Count}_L \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be the function given by

$$\mathrm{Count}_L(k) = \bigl|L \cap \Sigma^k\bigr|,$$

i.e., $\mathrm{Count}_L(k)$ is the number of strings of length k in Σ that are contained in the language. The point of this exercise is to show that if L is regular, then $\mathrm{Count}_L(k)$ must satisfy a number of restrictive conditions; to give some such conditions, we use linear algebra.

(1) Let L be a regular language recognized by a DFA with n states, $\{q_1, \ldots, q_n\}$, let M be the $n \times n$ matrix whose (i,j)-th entry is the number of symbols in Σ that take you from state q_i to state q_j. Describe the function $\mathrm{Count}_L(k)$ in terms of a sum of entries of M^k (the k-th power of M).

(2) Under the assumptions of Part (1), prove that for some $c_1, \ldots, c_n \in \mathbb{Z}$, $\mathrm{Count}_L(k)$ satisfies a recurrence equation for all $k \geq n$,

$$\mathrm{Count}_L(k) = c_1 \mathrm{Count}_L(k-1) + \cdots + c_n \mathrm{Count}(k-n).$$

(3) Consider the language over the alphabet $\Sigma = \{a\}$ given by

$$\text{UNARY-SQUARES} = \bigl\{a^{m^2} \,|m \in \mathbb{N}\bigr\} = \{a, a^4, a^9, \ldots\} = \{a, aaaa, aaaaaaaaa, \ldots\}.$$

Show that UNARY-SQUARES is not regular.

(4) Consider the language over the alphabet $\Sigma = \{a\}$ given by

$$L = \bigl\{a^m \in \{a\}^* \mid m \geq 20\bigr\}.$$

Show that any DFA recognizing L must have at least 21 states.

(5) Under the assumptions of Part (1), prove that if for reals a, b, C with $a, C > 0$ we have that as $k \to \infty$,

$$\mathrm{Count}_L(k) = C a^k k^b \bigl(1 + o(1)\bigr),$$

then b is an integer between 0 and $n-1$.

(6) Let

$$\text{PRIMES} = \bigl\{s \in \{0, 1, \ldots, 9\}^* \mid \text{in decimal, } s \text{ represents a prime number}\bigr\}$$

where leading 0's are not allowed. Show that PRIMES is not regular.

(7) Same question, where leading 0's are allowed.

Hints for Exercises Beyond: Hints appear on the page after this.

For Part (2), you might consider the Cayley-Hamilton theorem. For Part (5), the Perron-Frobenius theorem could simplify things; since M is not generally irreducible, you might want to write M as an upper triangular block matrix, where each diagonal block is irreducible.

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