HOMEWORK 1, CPSC 421/501, FALL 2019

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) Homework that is difficult to read may not be graded.
(4) You may work together on homework, **but you must write up your own solutions individually.** You must acknowledge with whom you worked. You must also acknowledge any sources you have used beyond the textbook and two articles on the class website.

In these exercises, "the handout" refers to the article "Self-referencing, Uncountability, and Uncomputability" on the 421/501 homepage.

(1) Consider the statement: "Alex cuts the hair of those (and only those) who do not cut their own hair." Is there a problem with this statement? Explain. To which "paradox" in the handout is this similar? Explain.

(2) Let W be the four element set

$$W = \{\texttt{one}, \texttt{two}, \texttt{plus}, \texttt{times}\}.$$

Ascribe a "meaning" to each sentence with words from W (i.e., each string over the alphabet W) in the usual way of evaluating expressions, so that

`one plus two times two` means $1 + 2 \times 2 = 5$,
`two times two times two` means $2 \times 2 \times 2 = 8$,

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and

`plus times two`	is meaningless,
`one plus two times`	is meaningless,
`one two`	is meaningless;

each sentence either "means" some positive integer or is "meaningless."

(a) Give two different sentences that both "mean" 10.
(b) Explain why every positive integer is the "meaning" of some sentence with words from W.
(c) Explain why every positive integer n is the "meaning" of some sentence of size at most $C(1+\log_2 n)^2$ for some constant $C \in \mathbb{R}$ independent of n; your explanation should give a value for C.

(3) Consider the five element set

$$U = W \cup \{\texttt{moo}\},$$

where W is the set in Exercise 2 with the ascribed meanings there, and where `moo` has the following meaning:

(a) if `moo` appears anywhere after the first word of a sentence, then the sentence is meaningless,
(b) if `moo` appears only once and at the beginning of a sentence, then we evaluate the rest of the sentence (as usual), and
    (i) if the rest evaluates to the integer k, then the sentence means "the smallest positive integer not described by a sentence of k words or fewer," and
    (ii) if the rest evaluates to meaningless, then the sentence is meaningless.

For example, "`moo moo`" and "`moo plus times two`" are meaningless, and "`moo two times two`" means "the smallest positive integer not described by a sentence of four words or fewer."

(a) What is the meaning of "`moo one`"?
(b) What is paradoxical in trying to ascribe a meaning to "`moo two`"?
(c) To which "paradox" in the handout is this similar? Explain.

(4) Which of the following maps are injections (i.e., one-to-one), and which are surjections (i.e., onto)? Briefly justify your answer.

(a) $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = x+1$.
(b) $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = x^2$.
(c) $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x+1$.
(d) $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$.

(5) If $f\colon S \to T$ and $g\colon T \to U$ are both injective (i.e., one-to-one), is $g \circ f$ (which is a map $S \to U$) necessarily injective? Justify your answer.

(6) Let $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, i.e.,

$$\mathbb{N}^2 = \{(n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\}.$$

(See Chapter 0 of [Sip].)

(a) Show that $\mathbb{N}^2$ is countable.
(b) Show that $\mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

(7) Let $S = \{a, b, c\}$ and let $f \colon S \to \mathrm{Power}(S)$ any function such that

$$a \notin f(a), \quad b \notin f(b), \quad c \notin f(c).$$

(a) Explain why $f(a)$ cannot be all of S.
(b) Explain why none of $f(a), f(b), f(c)$ equal S.
(c) What is the set

$$T = \{s \in S \mid s \notin f(s)\}?$$

(8) Let $S = \{a, b, c\}$ and let $f \colon S \to \mathrm{Power}(S)$ any function such that

$$a \notin f(a), \quad b \in f(b), \quad c \notin f(c).$$

(a) Explain why $f(b)$ cannot equal $\{a, c\}$.
(b) Explain why none of $f(a), f(c)$ equal $\{a, c\}$.
(c) What is the set

$$T = \{s \in S \mid s \notin f(s)\}?$$

(End of Homework Problems to be Submitted for Credit.)

Exercises Beyond the Homework (not for credit, solutions will not be released):

Cancellation Property: (1) We say that a map (of sets) $f\colon S \to T$ has the *left cancellation property* if for any two maps g, h from a set $U \to S$ we have $fg = fh$ (i.e., the map $f \circ g\colon U \to T$ equals the map $f \circ h$) implies that $g = h$. Show that this property holds of f iff f is injective.

(2) Formulate a similar *right cancellation property* for a map $f\colon S \to T$ and show that it is equivalent to f being surjective.

[This exercise shows that the notions of "injective" and "surjective" can be defined just in terms of sets and maps (also respectively called *objects* and *morphisms* (or *arrows*) in category theory).]

Unique Positive Rationals: Say that we list the positive rationals—allowing for repitition—as we did in class:

$$1/1, \quad 2/1,\ 1/2, \quad 3/1,\ 2/2,\ 1/3, \quad \ldots$$

Show that as $N \to \infty$, the number of distinct rational numbers in the first N terms of this sequence is

$$N\Bigl((1-1/4)\ (1-1/9)\ (1-1/25)\ (1-1/49)\ \ldots\Bigr) + o(N)$$

(i.e., $6N/\pi^2 + o(N)$, using a well-known value of the Riemann Zeta function).

[See the last page of this document for some hints; it is easier to see roughly why the above result is true than to give a rigorous proof of this result.]

Sample Exercises With Solutions:

People often ask me how much detail they need in giving explanations for the homework exercises. Here are some examples. The material in brackets [like this] are optional.

Sample Question Needing a Proof: If $f\colon S \to T$ and $g\colon T \to U$ are surjective (i.e., onto) is $g \circ f$ (a map $S \to U$) is necessarily surjective? Justify your answer.

Answer: Yes.

[To show that $g \circ f$ is surjective, we must show that if $u \in U$, then there is an $s \in S$ such that $(g \circ f)(s) = u$.]

If $u \in U$, then since g is surjective there is a $t \in T$ such that $g(t) = u$. Since f is surjective, there is an $s \in S$ such that $f(s) = t$. Hence

$$(g \circ f)(s) = g(f(s)) = g(t) = u.$$

Therefore each $u \in U$ is $g \circ f$ applied to some element of S, and so $g \circ f$ is surjective.

Sample Question Needing a Counterexample: If $f\colon S \to T$ is injective, and $g\colon T \to U$ is surjective, is $g \circ f$ is necessarily injective? Justify your answer.

Answer: No.

[To show that $g \circ f$ is not necessarily injective, we must find one example of such an f and g where $g \circ f$ is not injective.]

Let $S = T = \{a, b\}$ and $U = \{c\}$; let $f\colon S \to T$ be the identity map (i.e., $f(a) = a$ and $f(b) = b$), and let $g\colon T \to U$ (there is only one possible g in this case) be given by $g(a) = g(b) = c$.

Then f is injective (since $f(a) \neq f(b)$) and g is surjective, since $U = \{c\}$ and $c = g(a)$). However $g \circ f$ is not injective, since $(g \circ f)(a) = c = (g \circ f)(b)$.

Injectivitiy and Surjectivity of a Given Map: If $f\colon \mathbb{N} \to \mathbb{N}$ is given by $f(n) = 2n + 5$, is f injective? Is f surjective?

Answer: f is injective, because if $f(n_1) = f(n_2)$, then $2n_1 + 5 = 2n_2 + 5$ and therefore $n_1 = n_2$.

[Hence f maps distinct values of $\mathbb{N}$ to distinct values of $\mathbb{N}$, i.e., $n_1 \neq n_2$ implies that $f(n_1) \neq f(n_2)$.]

f is not surjective, because there is no value $n \in \mathbb{N}$ such that $f(n) = 1$: if such an n existed, then $2n + 5 = 1$ and so $n = -2$ which is not an element of $\mathbb{N}$.

Hints for Exercises Beyond: Hints appear on the page after this.

[Hint: It is easier to see why there should be roughly $6N/\pi^2$ distinct rationals in the first N terms than to give a rigorous proof of this result. For a rigorous proof, you could start by showing that the number of a/b in a sequence of length N where a, b are both divisible by two is (1) at most $N/4$ (with no $o(N)$ term) and (2) at least $N/4 + o(N)$. Then consider a, b which are either both divisible by two and/or both divisible by three. Etc. To give a rigorous proof of the $6N/\pi^2 + o(N)$ result you might use the fact that sum of $1/p^2$ over all prime numbers, p, converges, and hence the "infinite tails" of this sum tend to 0.]

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