THE UNIVERSITY OF BRITISH COLUMBIA
CPSC 303: MIDTERM EXAMINATION – March 15, 2024

Last Name: ______________________ First Name: ____________________

Signature: ______________________ UBC Student #: |_|_|_|_|_|_|_|_|

Important notes about this examination

1. You have 45 minutes to write this examination.
2. You may use a pencil to write your solutions, although a very light pencil might be harder to read after scanning.
3. No textbooks or electronic devices are permitted. We permit a "cheat-sheet" consisting of one page of handwritten or typed notes, on double-sided 8.5x11" paper.
4. Answer all the questions in the exam.
5. Good luck!

Student Conduct during Examinations

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0. Identification

Please make sure that the following is your 8-character Student ID:

`Student ID:`

Your answer to each problem should be written on its page; if needed, you can use the back side of the page as well.

1. Question 1. (10 points, 2 points per correct T/F Answer — No Penalty for Incorrect Responses)

Circle either T for true, or F for false, for each of the statements below.

Lagrange interpolation is numerically (i.e., in finite precision) more accurate than monomial interpolation when fitting data $(x_0, y_0), \ldots, (x_n, y_n)$ when the x_i are close together. **T** F

See Homework 7, Problem 3

The ODE $y' = |y|^{1/2}$ subject to $y(1) = 0$ has a unique solution $y = y(t)$ for near $t = 1$. T **F**

For any $a \leq b$, $y(t) = \begin{cases} -\frac{1}{4}(t-a)^2 & t \leq a \\ 0 & a \leq t \leq b \\ 1/4\,(t-b)^2 & t \geq b \end{cases}$ solves $y' = |y|^{1/2}$.

Hence if $y(1)=0$, both $y(t)=0$ near $t=1$ and $y(t) = \begin{cases} -1/4\, t^2 & t \leq 0 \\ 1/4\, t^2 & \geq 0 \end{cases}$

If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then the ODE $y' = f(y)$ has at least one solution $y = y(t)$ defined for all real t. T **F**

$y' = f(y)$ where $f(y)$ grows faster than linear will fail to have a solution; e.g. $y' = y^2 + 1$. *

If $A = SBS^{-1}$ for some square matrices A, B, S, then $A^{10} = SB^{10}S^{-1}$. **T** F

Yes $A^2 = (SBS^{-1})(SBS^{-1}) = SB^2S^{-1}$

$A^3 = A^2 A = (SB^2S^{-1})(SBS^{-1}) = SB^3S^{-1}$, etc.

(Recall that 2^{-1074} is the smallest positive subnormal number.) The expression $2^{-1074} \times 2^{1074}$ evalutes to 1 in MATLAB. T **F**

2^{1074} evaluates to Inf(inity); the largest normal number is $1.1\ldots1 \times 2^{1023}$; see also Homework 5, Problem 5(e).

And $y(t) = \begin{cases} -1/4\, t^2 & t \leq 0 \\ 1/4\, t^2 & \geq 0 \end{cases}$ has $y(t) \neq 0$ unless $t=0$.

* The example I had in mind was $y' = y^2$ or $y' = y^3$, as on the homework. But for these equations $y(t) = 0$ for all t works. One remedy is to make sure that $f(y) \geq$ some absolute constant. Or some other condition that forces $y(t)$ to reach any finite value for large enough t.

2. Question 2 (5 points)

Let $h > 0$ be a fixed, real number. Find the formula for x_n that solves the recurrence equation

$$x_{n+1} = (1+3h)x_n + 3h, \quad \text{for all integers } n,$$

subject to $x_0 = 1$. (Use the methods in class and on the homework.) Hence your formula for x_n should depend on h.

Homogeneous : $x_{n+1} - (1+3h)x_n = 0$

solution $x_n = (1+3h)^n C$ (C constant)

Special solution $x_{n+1} - (1+3h)x_n = 3h$

try $x_n = a$ (constant) :

$a - (1+3h)a = 3h \Rightarrow -3ha = 3h$

$\Rightarrow a = -1$

So special solution is $x_n = -1$

So " + homogeneous sol is

$x_n = (1+3h)^n C - 1$

$x_0 = 1 = (1+3h)^0 C - 1 = C - 1$ so $C = 2$

so $\boxed{x_n = 2(1+3h)^n - 1}$

3. Question 3 (5 points)

(1) Let m be a positive integer. Consider the ODE $y' = 3y + 3$ subject to $y(0) = 1$. What approximation does Euler's method give to $y(1)$ if you use step size $h = 1/m$? You may use your answer to Question 2 if you like.

(2) What is the limit as $m \to \infty$ of the approximation to $y(1)$ in part (1)?

[Hint 1: You can use the approximation $\log(1+\delta) = \delta + O(\delta^2)$ for $|\delta|$ near 0, or, equivalently, $1 + \delta = e^{\delta + O(\delta^2)}$.]

[Hint 2: The exact solution to $y' = 3y + 3$ subject to $y(0) = 1$ is $y(t) = 2e^{3t} - 1$. So your answer to part (2) should reflect this.]

(1) $y_{n+1} = y_n + h f(y_n) \qquad f(y) = 3y+3$

so

$$y_{n+1} = y_n + h(3y_n + 3)$$
$$= y_n(1+3h) + 3h.$$

by Question 2: $\quad y_n = 2(1+3h)^n - 1.$

If $h = \frac{1}{m}$, y_m approximates $y(mh) = y(1)$

Hence Euler approx to $y(1)$ is

$$\boxed{y_m = 2\left(1 + 3 \cdot \frac{1}{m}\right)^m - 1}$$

(2) $1 + 3\frac{1}{m} = e^{3/m + O(3/m)^2}$

$$\left(1 + 3\frac{1}{m}\right)^m = e^{[3/m + O(1/m^2)]m} = e^{3 + O(\frac{1}{m})}$$

So

$$\lim_{m\to\infty} 2\left(1+3\cdot\frac{1}{m}\right)^m - 1$$

$$= \lim_{m\to\infty} 2e^{3+O(1/m)} - 1$$

$$= 2e^3 - 1$$

(which agrees with exact solution).

4. Question 4 (5 points)

Let $p = p(x)$ and $q = q(x)$ be polynomials of degree at most 2 satisfying

$$p(1) = \sqrt{2},\ p(2) = \sqrt{3},\ p(3) = \sqrt{5},$$

and

$$q(11) = \sqrt{2},\ q(12) = \sqrt{3},\ q(13) = \sqrt{5}.$$

Show that $p(x) = q(x + 10)$ for all real x.

There are many solutions...

(1) Let $f(x) = p(x) - q(x+10)$, which is a polynomial of degree at most 2. But $f(1) = \sqrt{2} - \sqrt{2} = 0$, and similarly $f(2)=0$ and $f(3)=0$. Since f is of degree 2 and has 3 zeros, f is the zero polynomial. Hence $p(x) - q(x+10)$ is the zero polynomial, and so $p(x) = q(x+10)$. [See Homework 7, Problem (3)(e,f,g).]

(2) By Lagrange multipliers,

$$p(x) = \sqrt{2}\,\frac{(x-2)(x-3)}{(1-2)(1-3)} + \sqrt{3}\,\frac{(x-1)(x-3)}{(2-1)(2-3)} + \sqrt{5}\,\frac{(x-1)(x-2)}{(3-1)(3-2)}$$

$$q(x) = \sqrt{2}\,\frac{(x-12)(x-13)}{(11-12)(11-13)} + \sqrt{3}\,\frac{(x-11)(x-13)}{(12-11)(12-13)} + \sqrt{5}\,\frac{(x-11)(x-12)}{(13-11)(13-12)}$$

so

$$q(x+10) = \sqrt{2}\,\frac{(x+10-12)(x+11-12)}{(11-12)(11-13)} + \cdots$$

$$= \underbrace{\sqrt{2}\,\frac{(x-2)(x-3)}{(11-12)(11-13)}}_{=\ 1^{st}\text{ term in } p} + \underbrace{\sqrt{3}\text{ etc.}}_{\text{similarly } = 2^{nd}\text{ term in } p} + \underbrace{\sqrt{5}\text{ etc.}}_{\text{similarly } = 3^{rd}\text{ term in } p}$$

(3) We have $p(x) = c_0 + c_1 x + c_2 x^2$ where

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \\ \sqrt{5} \end{bmatrix}$$

If you now write $q(x) = \tilde{c}_0 + \tilde{c}_1 x + \tilde{c}_2 x^2$ where

$$\begin{bmatrix} 1 & 11 & 11^2 \\ 1 & 12 & 12^2 \\ 1 & 13 & 13^2 \end{bmatrix} \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \\ \sqrt{5} \end{bmatrix}$$

and the calculation of $c_0, c_1, c_2, \tilde{c}_0, \tilde{c}_1, \tilde{c}_2$ and of p, q is pretty long...

One clever idea I saw from some students is to write $q(x+10) = \hat{c}_0 + \hat{c}_1 x + \hat{c}_2 x^2$, and to note that

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \\ \sqrt{5} \end{bmatrix}$$

it follows that $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ and $\begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2 \end{bmatrix}$ must be equal

(we've seen that a system with a Vandermonde coefficient matrix has a unique solution).