

**THE UNIVERSITY OF BRITISH COLUMBIA**  
**CPSC 303: MIDTERM EXAMINATION – March 15, 2024**

Last Name: \_\_\_\_\_

First Name: \_\_\_\_\_

Signature: \_\_\_\_\_

UBC Student #: \_\_\_\_\_

**Important notes about this examination**

1. You have 45 minutes to write this examination.
2. You may use a pencil to write your solutions, although a very light pencil might be harder to read after scanning.
3. No textbooks or electronic devices are permitted. We permit a “cheat-sheet” consisting of one page of handwritten or typed notes, on double-sided 8.5x11” paper.
4. Answer all the questions in the exam.
5. Good luck!

**Student Conduct during Examinations**

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - i. speaking or communicating with other examination candidates, unless otherwise authorized;
  - ii. purposely exposing written papers to the view of other examination candidates or imaging devices;
  - iii. purposely viewing the written papers of other examination candidates;
  - iv. using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - v. using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).



## 0. IDENTIFICATION

Please make sure that the following is your 8-character Student ID:

**Student ID:**

**Your answer to each problem should be written on its page; if needed, you can use the back side of the page as well.**

1. QUESTION 1. (10 POINTS, 2 POINTS PER CORRECT T/F ANSWER — NO PENALTY FOR INCORRECT RESPONSES)

Circle either T for true, or F for false, for each of the statements below.

Lagrange interpolation is numerically (i.e., in finite precision) more accurate than monomial interpolation when fitting data  $(x_0, y_0), \dots, (x_n, y_n)$  when the  $x_i$  are close together.

T  F

See Homework 7, Problem 3

The ODE  $y' = |y|^{1/2}$  subject to  $y(1) = 0$  has a unique solution  $y = y(t)$  for near  $t = 1$ .

For any  $a \leq b$ ,  $y(t) = \begin{cases} -\frac{1}{4}(t-a)^2 & t \leq a \\ 0 & a \leq t \leq b \\ \frac{1}{4}(t-b)^2 & t \geq b \end{cases}$  solves  $y' = |y|^{1/2}$ .

T  F

Hence if  $y(1) = 0$ , both  $y(t) = 0$  near  $t = 1$  and  $y(t) = \begin{cases} -\frac{1}{4}t^2 & t \leq 0 \\ \frac{1}{4}t^2 & t \geq 0 \end{cases}$

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is differentiable, then the ODE  $y' = f(y)$  has at least one solution  $y = y(t)$  defined for all real  $t$ .

T  F

$y' = f(y)$  where  $f(y)$  grows faster than linear will fail to have a solution; e.g.  $y' = y^2 + 1$ . \*

If  $A = SBS^{-1}$  for some square matrices  $A, B, S$ , then  $A^{10} = SB^{10}S^{-1}$ .

T  F

Yes  $A^2 = (SBS^{-1})(SBS^{-1}) = SB^2S^{-1}$

$A^3 = A^2 A = (SB^2S^{-1})(SBS^{-1}) = SB^3S^{-1}$ , etc.

(Recall that  $2^{-1074}$  is the smallest positive subnormal number.) The expression  $2^{-1074} \times 2^{1074}$  evaluates to 1 in MATLAB.

T  F

$2^{1074}$  evaluates to Inf(infinity); the largest normal number is  $1.1...1 \times 2^{1023}$ ; see also Homework 5, Problem 5(e).

And  $y(t) = \begin{cases} -\frac{1}{4}t^2 & t \leq 0 \\ \frac{1}{4}t^2 & t \geq 0 \end{cases}$  has  $y(t) \neq 0$  unless  $t = 0$ .

\* The example I had in mind was  $y' = y^2$  or  $y' = y^3$ , as on the homework. But for these equations  $y(t) = 0$  for all  $t$  works. One remedy is to make sure that  $f(y) \geq$  some absolute constant. Or some other condition that forces  $y(t)$  to reach any finite value for large enough  $t$ .

## 2. QUESTION 2 (5 POINTS)

Let  $h > 0$  be a fixed, real number. Find the formula for  $x_n$  that solves the recurrence equation

$$x_{n+1} = (1 + 3h)x_n + 3h, \quad \text{for all integers } n,$$

subject to  $x_0 = 1$ . (Use the methods in class and on the homework.) Hence your formula for  $x_n$  should depend on  $h$ .

Homogeneous :  $x_{n+1} - (1+3h)x_n = 0$

solution  $x_n = (1+3h)^n C$  ( $C$  constant)

Special solution  $x_{n+1} - (1+3h)x_n = 3h$

try  $x_n = a$  (constant) :

$$a - (1+3h)a = 3h \quad \Rightarrow \quad -3ha = 3h$$

$$\Rightarrow a = -1$$

So special solution is  $x_n = -1$

So " + homogeneous sol is

$$x_n = (1+3h)^n C - 1$$

$$x_0 = 1 = (1+3h)^0 C - 1 = C - 1 \quad \text{so } C = 2$$

so

$$x_n = 2(1+3h)^n - 1$$

## 3. QUESTION 3 (5 POINTS)

- (1) Let  $m$  be a positive integer. Consider the ODE  $y' = 3y + 3$  subject to  $y(0) = 1$ .  
 What approximation does Euler's method give to  $y(1)$  if you use step size  $h = 1/m$ ?  
 You may use your answer to Question 2 if you like.
- (2) What is the limit as  $m \rightarrow \infty$  of the approximation to  $y(1)$  in part (1)?

[Hint 1: You can use the approximation  $\log(1+\delta) = \delta + O(\delta^2)$  for  $|\delta|$  near 0, or, equivalently,  $1 + \delta = e^{\delta + O(\delta^2)}$ .]

[Hint 2: The exact solution to  $y' = 3y + 3$  subject to  $y(0) = 1$  is  $y(t) = 2e^{3t} - 1$ . So your answer to part (2) should reflect this.]

$$(1) \quad y_{n+1} = y_n + h f(y_n) \quad f(y) = 3y + 3$$

so

$$\begin{aligned} y_{n+1} &= y_n + h(3y_n + 3) \\ &= y_n(1 + 3h) + 3h. \end{aligned}$$

by Question 2 :  $y_n = 2(1 + 3h)^n - 1$ .

If  $h = \frac{1}{m}$ ,  $y_m$  approximates  $y(mh) = y(1)$

Hence Euler approx to  $y(1)$  is

$$y_m = 2 \left(1 + 3 \cdot \frac{1}{m}\right)^m - 1$$

$$(2) \quad 1 + 3 \frac{1}{m} = e^{3/m + O(3/m)^2}$$

$$\left(1 + 3 \frac{1}{m}\right)^m = e^{[3/m + O(1/m^2)]m} = e^{3 + O(1/m)}$$

S<sub>0</sub>

$$\lim_{m \rightarrow \infty} 2 \left(1 + 3 \cdot \frac{1}{m}\right)^m - 1$$

$$= \lim_{m \rightarrow \infty} 2 e^{3 + o(1/m)} - 1$$

$$= 2 e^3 - 1$$

(which agrees with exact solution).

## 4. QUESTION 4 (5 POINTS)

Let  $p = p(x)$  and  $q = q(x)$  be polynomials of degree at most 2 satisfying

$$p(1) = \sqrt{2}, p(2) = \sqrt{3}, p(3) = \sqrt{5},$$

and

$$q(11) = \sqrt{2}, q(12) = \sqrt{3}, q(13) = \sqrt{5}.$$

Show that  $p(x) = q(x+10)$  for all real  $x$ .

There are many solutions...

① Let  $f(x) = p(x) - q(x+10)$ , which is a polynomial of degree at most 2. But  $f(1) = \sqrt{2} - \sqrt{2} = 0$ , and similarly  $f(2) = 0$  and  $f(3) = 0$ . Since  $f$  is of degree 2 and has 3 zeros,  $f$  is the zero polynomial. Hence  $p(x) - q(x+10)$  is the zero polynomial, and so  $p(x) = q(x+10)$ . [See Homework 7, Problem (3)(e,f,g).]

② By Lagrange multipliers,

$$p(x) = \sqrt{2} \frac{(x-2)(x-3)}{(1-2)(1-3)} + \sqrt{3} \frac{(x-1)(x-3)}{(2-1)(2-3)} + \sqrt{5} \frac{(x-1)(x-2)}{(3-1)(3-2)}$$

$$q(x) = \sqrt{2} \frac{(x-12)(x-13)}{(11-12)(11-13)} + \sqrt{3} \frac{(x-11)(x-13)}{(12-11)(12-13)} + \sqrt{5} \frac{(x-11)(x-12)}{(13-11)(13-12)}$$

so

$$q(x+10) = \sqrt{2} \frac{(x+10-12)(x+10-13)}{(11-12)(11-13)} + \dots$$

$$= \underbrace{\sqrt{2} \frac{(x-2)(x-3)}{(11-12)(11-13)}}_{= 1^{\text{st}} \text{ term in } p} + \underbrace{\sqrt{3} \text{ etc.}}_{= 2^{\text{nd}} \text{ term in } p} + \underbrace{\sqrt{5} \text{ etc.}}_{= 3^{\text{rd}} \text{ term in } p}$$

③ We have  $p(x) = c_0 + c_1x + c_2x^2$  where

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \\ \sqrt{5} \end{bmatrix}$$

If you now write  $q(x) = \tilde{c}_0 + \tilde{c}_1x + \tilde{c}_2x^2$  where

$$\begin{bmatrix} 1 & 11 & 11^2 \\ 1 & 12 & 12^2 \\ 1 & 13 & 13^2 \end{bmatrix} \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \\ \sqrt{5} \end{bmatrix}$$

and the calculation of  $c_0, c_1, c_2, \tilde{c}_0, \tilde{c}_1, \tilde{c}_2$  and of  $p, q$  is pretty long...

One clever idea I saw from some students is to write

$$q(x+10) = \hat{c}_0 + \hat{c}_1x + \hat{c}_2x^2, \text{ and to note that}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \\ \sqrt{5} \end{bmatrix}$$

it follows that  $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$  and  $\begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2 \end{bmatrix}$  must be equal

(we've seen that a system with a Vandermonde coefficient matrix has a unique solution).