CPSC 303, April 12, 2024

- Review ODE schemes:
  - Where do they come from?
  - How accurate are they?

- The stiff equation

$$\vec{Y}' = \begin{bmatrix} -1000 & 0 \\ 0 & 1 \end{bmatrix} \vec{Y}$$

Bad case of $\vec{Y}' = A\vec{Y}$

- Complex eigenvalues w/o complex eigenvalues: why is

$$\ddot{u} = -c^2 u$$

"stiff"?

- What about the central force problem?

Note:

- Additional office hours next week.

- More practice final questions to appear early next week (with brief solutions).

OUR ODE SCHEMES:

For $y' = f(y)$, $y(0) = y_0$:

Forward Euler:

$$y_{i+1} = y_i + h f(y_i)$$

Backward Euler: ← (for stiff equations)

$$y_{i+1} = y_i + h f(y_{i+1})$$

← implicit

Implicit Trapezoidal:

$$y_{i+1} = y_i + h \frac{f(y_i) + f(y_{i+1})}{2}$$

← 2nd order scheme

Explicit Trapezoidal

2nd order scheme

$$y_{i+1} = y_i + h\,\frac{f(y_i) + f(Y_i)}{2}$$

(solved $y' = ay$ J)

where $$Y_i = y_i + h\,f(y_i)$$

RK4 (Runga-Kutta 4)

$$Y_{i,1} = y_i, \quad Y_{i,2} = y_i + \frac{h}{2} f(Y_1)$$

$$Y_{i,3},\ Y_{i,4} = \text{blah blah blah}$$

$$y_{i+1} = y_i + h\,(\text{blah blah blah})$$

Where do they come from?

Forward Euler is direct,

others are more puzzling...

$$y' = f(y), \qquad y(0) = y_0 \in \mathbb{R}$$

Idea! choose $h > 0$ "small",

$t_0 = 0, \quad t_1 = h, \quad t_2 = 2h, \quad t_3 \ldots$

$$y(t+h) = y(t) + h\,y'(t) + \frac{h^2}{2} y''(t) + \ldots$$

$$= y(t) + h\,y'(t) + O(h^2)$$

$$t_{i+1} = t_i + h$$

$$y(t_{i+1}) = y(t_i) + h\,y'(t_i) + \underbrace{O(h^2)}_{\text{drop}}$$

$$y(t_{i+1}) \approx y(t_i) + h\,y'(t_i)$$

$$= y(t_i) + h\,f(y(t_i))$$

$$y_{i+1} = y_i + h\,f(y_i)$$

Error: hopefully ...

Since

$$y_{i+1} - y_i + h f(y_i) = O(h^2)$$

hopefully

$$\text{error}_i = y_{i+1} - y_i + h f(y_i) + O(h^2)$$

y_{i+1} given y_i $\quad O(h^2)$

y_i given y_{i-1} $\quad O(h^2)$

$\vdots$

$\sum$ errors : $\quad O(i h^2)$

$$Y(T) = y(t_i) = y(ih)$$

Error

$$y(T) - y_i$$

(where $ih = T$, $i = T/h$)

is just

$$i \; O(h^2)$$

$$T/h \cdot O(h^2)$$

$$= O(T \cdot h)$$

truth!

kinda true, but

constant y'',

Lipschitz $f(y)$, . . —

For practical considerations:

$$y' = a y \qquad y(0) = y_0$$

$$\vec{y}\,' = A \vec{y} \qquad \vec{y}(0) = \vec{y}_0 .$$

$$y_1 = y(h) = e^{ah} y_0$$

$$= \left(1 + ah + \frac{(ah)^2}{2} + \frac{(ah)^3}{3!} + \dots \right) y_0$$

Forward Euler:

$$y' = ay, \qquad y_{i+1} = y_i + h f(y_i)$$

$$= y_i(1+ha)$$

$$y_{i+1} = (1+ha)^i y_0$$

Fix $T > 0$

$$y_{T/h} = \underbrace{(1+ah)^{T/h}}_{\xrightarrow{h \to 0} e^{aT}} y_0$$

What could possibly go wrong?

Consider:

Say:

"Think of this as a black box"

$$\vec{y}\,' = \begin{bmatrix} -1000 & 0 \\ 0 & 2 \end{bmatrix} \vec{y}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{bmatrix} -1000\, y_1 \\ 2\, y_2 \end{bmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$y_1(t) = e^{-1000t}\, y_1(0) = e^{-1000t}\, 2,$$

$$y_2(t) = e^{2t}\, y_2(0) = e^{2t}\, 3.$$

Say $t=1$:

$$\vec{y}(t) = \left(e^{-1000t}\,2,\ e^{2t}\,3\right)\Big|_{t=1}$$

$$= \left(e^{-1000}\,2,\ e^{2}\cdot 3\right)$$

looks negligible

But Euler's method:

$$\vec{y}_{i+1} = (I + hA)\,\vec{y}_i$$

$$\vec{Y}_i = (I + hA)^i \, \vec{Y}_0$$

$$Y(1) \approx Y_{1/h}$$

$$Y_{1/h} = (I + hA)^{1/h} \, \vec{Y}_0$$

$$= \begin{bmatrix} 1 + h(-1000) & 0 \\ 0 & 1 + 2h \end{bmatrix}^i \vec{Y}_0$$

$$= \begin{bmatrix} (1 + h(-1000))^{1/h} & \\ & (1+2h)^{1/h} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So $h = \frac{1}{5}, \frac{1}{10}, \frac{1}{20},$

this works for

$$y_2' = 2y_2$$

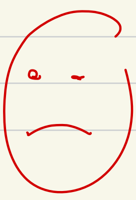
$$Y_{2,1/h} \approx (1+2h)^{1/h}\, 3$$

But for even $h = \frac{3}{1000}$,

So $ah = \frac{3}{1000}(-1000) = -3$

$$1+ah = -2$$

$$(1+ah)^{1/h} = (-2)^{1/h}$$



Fix! Backward Euler:

$$y_{i+1} = y_i + h\, f(\underbrace{y_{i+1}}_{})$$

still $O(h^2)$ error $\quad y_i \to y_{i+1}$

overall global error $O(Th)$

first order scheme

$$y_{i+1} = y_i + h\,(a\, y_{i+1})$$

$$y_{i+1}(1-ha) = y_i$$

$$y_{i+1} = y_i / (1-ah)$$

But you really have to tailor the scheme when problems arise.

Backwards Euler solves

eigenvalue$_1$ = -1000

$\vdots$ } "small"

$$\begin{bmatrix} -1000 & 0 \\ 0 & 2 \end{bmatrix} = \mathrm{diag}(-1000, 2)$$

eigenvalues $-1000, 2$

Complex eigenvalues :

$$\ddot{x} = -c^2 x$$

solutions

$$x(t) = c_1 \sin(ct) + c_2 \cos(ct)$$

c big $\Rightarrow$ oscillating more quickly

$$y = \begin{bmatrix} \dot{x} \\ cx \end{bmatrix}$$

$$\dot{y} = \begin{bmatrix} \ddot{x} \\ c\dot{x} \end{bmatrix} = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ cx \end{bmatrix}$$

$c = 1$

$$\dot{\vec{y}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{y}$$

Euler's method:

$$\vec{y}_{i+1} = \left(I + hA\right) \vec{y}_i$$

$$= \begin{bmatrix} 1 & -h \\ h & 1 \end{bmatrix} \vec{y}_i$$

c

Energy!

$$(\dot{x})^2 + (cx)^2$$

$c = 1$

$$E(t) = (\dot{x})^2 + x^2$$

$$= (\dot{x}(t))^2 + (x(t))^2$$

$$\frac{d}{dt} E(t) = 0 \Rightarrow E(t) \text{ constant}$$

$$y = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}, \quad E = \|y\|^2$$

Any $\quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$(I + hA)\vec{z}$$

$$\begin{bmatrix} 1 & -h \\ h & 1 \end{bmatrix} \vec{z} = \begin{bmatrix} z_1 - h z_2 \\ z_2 + h z_1 \end{bmatrix}$$

$$\left\| \begin{bmatrix} \quad \end{bmatrix} \right\| = (z_1 - h z_2)^2 + (z_1 + h z_2)^2$$

$$= (z_1^2 + z_2^2)(1 + h^2)$$

So

$$\text{energy}(y_i)$$

$$= (1+h^2)^i \, \text{energy}(y_0)$$

}, if $\ddot{x} = c^2 x$

},

$$(1+c^2h^2)^i$$