April 10, 2024

Today: Stiff Equations,

Implicit Euler's method. (Sections 10.2, 10.3 of [A&G])

There is a

Final Exam Study Guide

- Outline: Sections of [A&G], classes, ...

- Skills: a long list

- Sample Final Exam Problems (mostly on stuff after midterm)

Last time:

ODE $\quad \vec{y}\,' = A\vec{y}$

or $\quad y' = ay \quad a \in \mathbb{R}$

Euler's method

$$\vec{y}(t_0) = \vec{y}_0$$

$h \quad t_1$

$h \quad t_2$

$\vdots$

$\vec{y}_m$ approx $\vec{y}(t_0 + hm)$

$$\vec{y}_{m+1} = \vec{y}_m + h f(\vec{y}_m)$$

$$= (I + hA) \vec{y}_m$$

for $\vec{y}\,' = A \vec{y}$

$$\vec{y}_m = (I + hA)^m \; \vec{y}_0$$

(For heat eq. analog,

$$(I + hA)^m \rightsquigarrow$$

$$\left((1-2\rho) I + \rho(\cos(\pi h) - 1) \right)^m$$

in case $u(x,0) = \sin(\pi x)$)

Say $y' = ay$

or --- $y' = \lambda y$

so $y(0) = y_0$

$$Y_m = (1 + \lambda h)^m \, y_0$$

$y(T)$, fix T, let $h \to 0$

$$Y_m \approx y(mh) \leftrightarrow y(T)$$

$m = T/h$

so

$y(T)$ approx by

$$y_{T/h} = (1+\lambda h)^{T/h} \, y(0)$$

$$\shortparallel$$

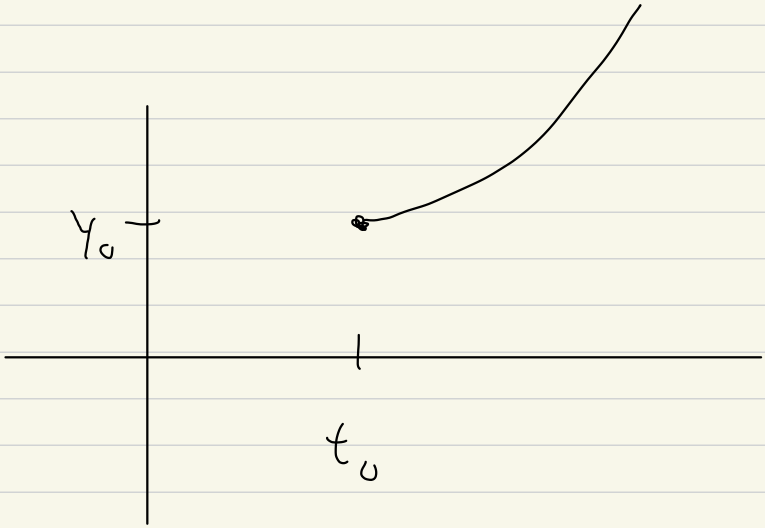
$$y_m$$

$$\lim_{h \to 0} (1+\lambda h)^{T/h} = e^{\lambda T}$$

What goes wrong ...

If $\lambda \geq 0$, then

$$y(t) = e^{\lambda(t-t_0)}\, y(t_0)$$



$\lambda > 0$

approx: $(1+\lambda h)^{T/h}$ increases

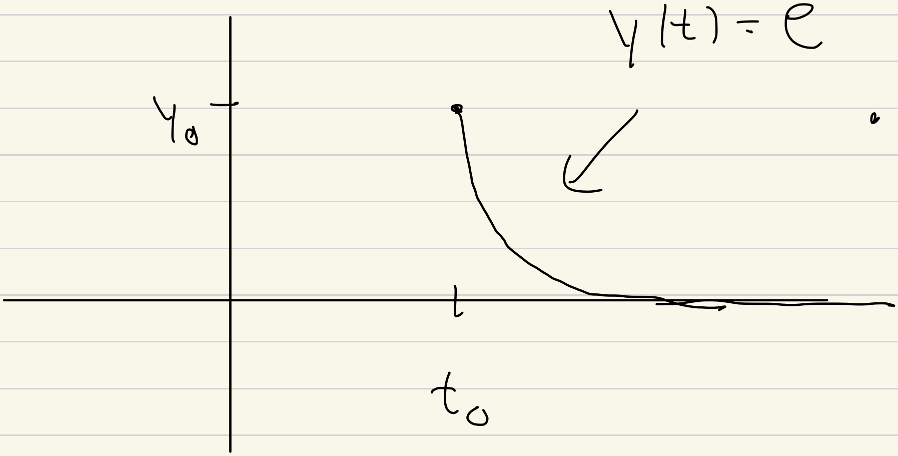
undershoots:

$$e^{\lambda h} = 1 + \lambda h + \frac{(\lambda h)^2}{2} + \dots$$

Problem: what if $\lambda < 0$

Example $\lambda = -1000$

$y(t) = e^{-1000(t-t_0)} \cdot y_0$



but

$$y_m = \underbrace{(1+\lambda h)^{T/h}}_{(1-1000h)^{T/h}} \; y_0$$

If $|1-1000h|$ not ≤ 1

If $h = \frac{1}{100}$, then

$$y_m = \left(1 - 1000 \cdot \frac{1}{100} \right)^m y_0$$

$$= (1 - 10)^m y_0$$

$$= (-9)^m y_0$$

What to do ---

Points:

(1) You have to modify something in $\lambda = -1000$

(2) If $\vec{y}' = A\vec{y}$, $\vec{y}' = \vec{f}(\vec{y})$

you have $\lambda = -1000$, but it's a bit hidden...

(3) If $\lambda \in \mathbb{C}$, λ not real, similar issues, but worse...

"Stiff Equation" refers to

- λ very negative, or
- A with very negative eigenvalues, or
- $\vec{y}\,' = \vec{f}(\vec{y})$ where near $\vec{y}_0$,

$$\vec{f}(\vec{y}) \approx \underbrace{\vec{c}_0}_{\text{const}} + \underbrace{A(\vec{y}-\vec{y}_0)}_{\text{linear}}$$

+ error + Quadratic

Solution to λ very negative

or $\vec{y}\,' = A\vec{y}$ where

eigenvalues of A very

negative

$$A = S \begin{bmatrix} \lambda_1 & & 0's \\ & \lambda_2 & \\ 0's & & \ddots & \\ & & & \lambda_n \end{bmatrix} S^{-1}$$

$$A^k = S \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}^k S^{-1}$$

$$= S \begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \ddots \end{bmatrix} S^{-1}$$

Eulers method:

$$y(t+h) \approx y(t) + h\,y'(t)$$

$$(y'(t) = \lambda y) \quad \approx y(t) + h\lambda y(t)$$

$$y_{m+1} = y_m + h\,f(y_m)$$

$$= y_m(1 + h\lambda)$$

$$\frac{y(t+h) - y(t)}{h} = y'(t) + O(h)$$

$$= y'\left(t + \frac{h}{2}\right) + O(h^2)$$

$$= y'(t+h) + O(h)$$

So:

$$y(t+h) = y(t) + h\, f(y(t+h))$$

Implicit Euler's Method

Backward " "

$$\boxed{y_{m+1} = y_m + h\, f(y_{m+1})}$$

if $y' = \lambda y$

$$\boxed{y_{m+1} = y_m + h\, y_{m+1}}$$

Implicit because

$$y_{m+1} = y_m + h\, f(y_{m+1})$$

? known ?

(Rem: If h suff small:

$$y_{m+1} = y_m + h\, f(y_{m+1})$$

$$= y_{m+1} + h\, f(y_m + h\, y_{m+1})$$

$y_m +$

$h\, f(y_{m+1})$

Imagine we can find y_{m+1} from

$$y_{m+1} = y_m + h\, f(y_{m+1})$$

then $\quad y' = -1000\, y$

$$y_{m+1} = y_m + h\left(y_{m+1}(-1000)\right)$$

$$y_{m+1}(1 + 1000h) = y_m$$

$$y_{m+1} = \frac{y_m}{1 + 1000\, h}$$

Then $y' = \lambda y$ you get

$$y_{m+1}(1 - \lambda h) = y_m$$

$$y_{m+1} = \frac{y_m}{(1 - \lambda h)}$$

$$\lambda = 0, -1, -10, -1000, \ldots$$

So $\lambda < 0$, y_m decreases

and is qualitatively better...

But if λ positive: $\frac{1}{1-\lambda h}$

If $\frac{1}{1-\lambda h}$ is negative

$$1 - \lambda h < 0$$

$$1 < \lambda h \qquad h > \frac{1}{\lambda}$$

So

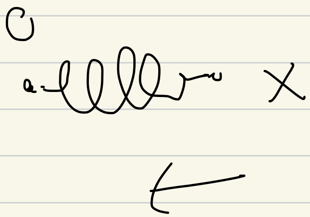
$$\vec{y}\,' = \begin{bmatrix} 1000 & 0 \\ 0 & -1000 \end{bmatrix} \vec{y}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1000\, y_1 \\ -1000\, y_2 \end{pmatrix}$$

then you need something else...

Consider

$$x'' = -c^2 x$$



We know

$$C_1 \sin(ct) + C_2 \cos(ct)$$

gives a solution $x(t)$.

Also

$$\overline{Y} = \begin{bmatrix} x' \\ x \end{bmatrix}$$

$$\vec{Y}' = \begin{bmatrix} x' \\ x \end{bmatrix}' = \begin{bmatrix} x'' \\ x' \end{bmatrix}$$

$$= \begin{bmatrix} -c^2 x \\ x' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ x \end{bmatrix}$$

$$= \underbrace{A}_{\begin{bmatrix} 0 & -c^2 \\ 1 & 0 \end{bmatrix}} \vec{Y}$$

Note!

given $\vec{y}(0) = \begin{bmatrix} x'(0) \\ x(0) \end{bmatrix}$

$$x(t) = C_1 \sin(ct) + C_2 \cos(ct)$$

stay bounded ...

So ...

$$\vec{y}_m = \underbrace{(1 + Ah)^m}_{\text{want to be bounded}} \vec{y}_0$$

What is

$$(I + Ah)^m \quad \text{in}$$

size?

Find eigenvalues of $\begin{bmatrix} 0 & -c^2 \\ 1 & 0 \end{bmatrix}$;

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -c^2 \\ 1 & 0 \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} \lambda & c^2 \\ -1 & \lambda \end{bmatrix} = 0$$

$$\lambda^2 + c^2 = 0$$

complex number

$$\lambda = \pm \sqrt{-1}\ c$$

$$(1+h\lambda)^m$$

$$= \left(1 \pm h\sqrt{-1}\,c\right)^m$$

$$(1 \pm hic)^m$$

$$\left|(1+ihc)\right|^m$$

$$= \sqrt{1+(hc)^2}^{\,m}$$

if hc is any fixed, pos real

$$\left(1 + (hc)^2\right)^{m/2}$$

$$\approx \left(\begin{array}{c}\text{pos number} \\ > 1\end{array}\right)^{m/2}$$

Real case!

$$(1 - 1000h)^m$$

So

$$-1 < (1 - 1000h) < 1$$

$$2 < -1000h < 0$$