CPSC 303, April 8, 2024

How do we approach ODE's and PDE's

Heat Eq...

ODE's:

$$\vec{y}\,' = f(t, \vec{y}) :$$

(1) Look locally:

$$\vec{y}\,' = \vec{f}(t, \vec{y}), \quad \vec{y}(t_0) = \vec{y}_0$$

t near t_0, Taylor series, ...

Euler's Method, Explicit

Trapezoidal, ...

(2) Real Question: Do these methods work for t near $t_1 \neq t_0$, step size, h, in Euler's method, Trapezoidal, ... $h \to 0$ and approximate $y(t_1)$

Method: Choose a simple ODE, for which we do know solution

$$\vec{y}\,'(t) = A\vec{y}(t)$$

we knew

$$\vec{y}(t) = e^{A(t-t_0)}\vec{y}_0$$

if we impose $\vec{y}(t_0) = \vec{y}_0$.

<u>Method 2</u>: If $\vec{y}' = \vec{f}(t, \vec{y})$

$\Rightarrow$ some invariant is preserved (Energy), then you can run the method and test if the preserved invariant is numerically preserved.

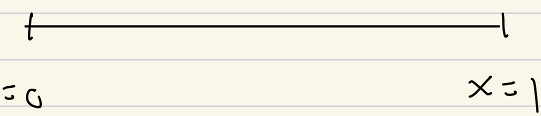
<u>Method 2'</u> If $\vec{y}' = \vec{f}(t, \vec{y})$ can be "reversed in time"

then you can run the equations, reverse them, and you should get back to your initial conditions

Still, things can go wrong ...

Do the same thing for the heat equation:

$$u_t = u_{xx}$$



We designed a method

$$u_t \mathrel{\hat{=}} \frac{u(x,t+H) - u(x,t)}{H}$$

$$u_{xx} \cong \frac{u(x+h) + u(x-h) - 2u(x)}{h^2}$$

$u(x, t+H)$

$= u(x,t)(1-2\rho) +$

$(u(x+h,t) + u(x-h,t))\ \rho$

$$\rho = \frac{h^2}{H} \qquad \left(\text{or } \frac{ch^2}{H} \text{ for } u_t = c\,u_{xx} \right)$$

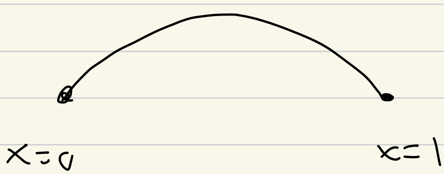
Rem! If $\rho = 1/6$, you get a scheme that is accurate ~~to~~ to higher order

Method: Find a simple, exact solution, and see how things work

Simple solution:

$$u(x,t) = \sin(\pi x)\, e^{-\pi^2 t}$$

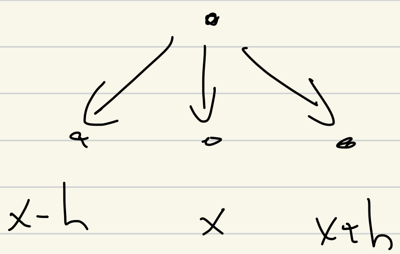
$t = 0$ $\quad u(x,0) =$



$\sin(\pi x)$

$t > 0$ the solution decays by $e^{-\pi^2 t}$

Claim! We can test scheme



$$u(x, t+H) \simeq (1-2\rho)\, u(x,t) + \rho \big(u(x-h,t) + u(x+h,t) \big)$$

$$\boxed{u(x,t) = \sin(\pi x) \cdot e^{-\pi^2 t}}$$

$$= e^{-\pi^2 t} \cdot \Big((1-2\rho) \sin(\pi x) + \rho \big[\sin(\pi(x-h)) + \sin(\pi(x+h)) \big] \Big)$$

$$\sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha \cos\beta - \quad " \quad "$$

$$(\sin\alpha)(2\cos\beta)$$

$$\sin(\pi x - \pi h)$$
$$+$$
$$\sin(\pi x + \pi h) = (\sin(\pi x))(2\cos(\pi h))$$

So!

$$(1-2p)\sin(\pi x) + p(\sin(\pi(x-h)) + \sin(\pi(x+h)))$$

$$=$$

$$(\sin(\pi x))(1-2p+p\,2\cos(\pi h))$$

[Analog of $y' = Ay \ldots$]

Eulers method:

$$y(t+h) \approx y(t) + h\, f(t,y)$$
$$h A y(t)$$

$$y_{m+1} = y_m + hA\, y_m$$
$$= (1+hA)\, y_m$$

So
$$y_m = (1+hA)^m\, y_0$$

So

$$u(x,t+H) = (1-2\rho)\,u(x,t) + \rho \left[\begin{array}{l} u(x+h,t) + \\ u(x-h,t \end{array}\right]$$

In case
$$u(x,0) = \sin(\pi \cancel{x})$$
$$u(x,t) = \sin(\pi x)\, e^{-\pi^2 t}$$

$$u(x,t+H) = u(x,t)\left(1-2\rho+2\rho\cos(\pi h)\right)$$

$$u(x,0) = \sin(\pi x)$$

$$u(x,H) \approx \left(1-2\rho+2\rho\cos(\pi h)\right) u(x,0)$$

$u(x,2H) \approx$ in our heat eq approx, in exact arithmetic

$$= \left(1-2\rho+2\rho\cos(\pi h)\right)^2 u(x,0)$$

$\vdots$

$$u(x,sH) = \left(1-2\rho+2\rho\cos(\pi h)\right)^s u(x,0)$$

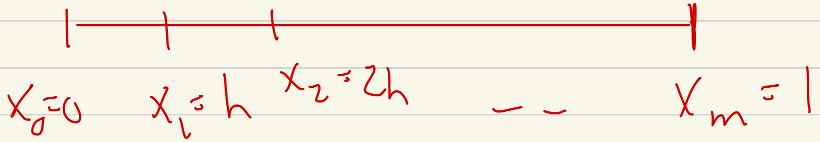
Fix ρ, take $h \to 0$, $\rho = \frac{h^2}{H}$, $H = \frac{h^2}{\rho}$

$$u\left(x, s\frac{h^2}{\rho}\right) \approx \left(1-2\rho+2\rho\cos(\pi h)\right)^s u(x,0)$$

Say t_1,

assume ~~take~~ h small with $s\frac{h^2}{\rho} = t_1$

for some $s \in \mathbb{N}$, $h = \frac{1}{m}$



$= hm$

$h = 1/m$

$$\lim_{h \to 0} \left(1 - 2\rho + 2\rho \cos(\pi h)\right)^s$$

$$\longrightarrow e^{-\pi^2 t_1}$$

(Analog: Euler's method $y' = Ay$:)

$$y_n = (1+Ah)^n y_0$$

$$\lim_{h\to 0} (1+Ah)^n = e^{At_1}$$

$$nh = t_1 - t_0$$

Trap. $y_n = \left(1 + Ah \pm \frac{h^2 A}{2} ?\right)^n y_0$

$$\cos(\pi h) \simeq 1 - \frac{(\pi h)^2}{2!} + \frac{(\pi h)^4}{4!} - \cdots$$

So

$$1 - 2\rho + 2\rho\cos(\pi h)$$

$$= 1 - 2\rho + 2\rho\left(1 - \frac{(\pi h)^2}{2!} + O(h^4)\right)$$

$$\approx 1 - \frac{(\pi h)^2}{2} + O(h^4)$$

So

$$\left(1 - \frac{(\pi h)^2}{2} + O(h^4)\right)$$

need something order $1/h^2$

So time ~~step~~ step size

H must be roughly order h^2

So just as Trapezoidal rule converges faster than Euler's method for $y' = Ay$, the calculation

$$u(x, Hs) = \left(1 - 2\rho + 2\rho\cos(\pi h)\right)^{S} u(x,0)$$

for $u(x,0) = \sin(\pi x)$

You can test that for $\rho = 1/6$,

this convergence ($h \to 0$, $H \approx \rho h^2$)

is faster by a order in h^2

$$1 - 2\rho + 2\rho\cos(\pi h)$$

$$= 1 - 2\rho\left(\frac{(\pi h)^2}{2!} + \frac{(\pi h)^4}{4!} - \cdots\right)$$