CPSC 303, April 5, 2024:

The heat equation $u_t = c\, u_{xx}$

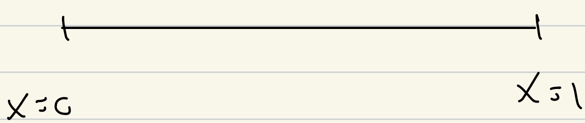
and $N_{rod,n} = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}$

Goal:

- Explain basic numerical scheme
- Explain when the numerical scheme becomes unstable ($\rho > 1/2$)

Heat equation:

1-dim idealized rod, wire



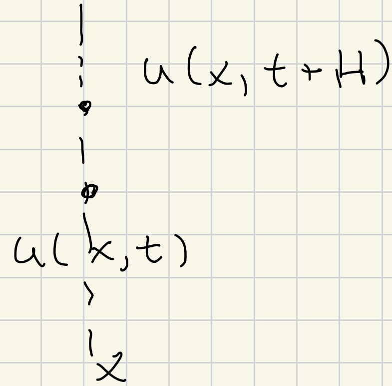
$u(x,t)$ = temperature of the rod in position x, $0 \leq x \leq l$, time $t \geq 0$

Equation:

$$u_t = c\, u_{xx}$$

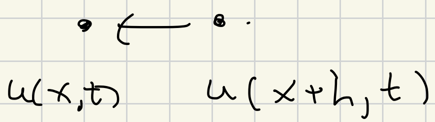
c constant.

$u_t = u_t(x,t)$ is the first derivative in t of $u(x,t)$, viewing x as fixed:



$$u_t(x,t) = \lim_{H \to 0} \frac{u(x,t+H) - u(x,t)}{H}$$

$$u_x(x,t) = \lim_{h \to 0} \frac{u(x+h,t) - u(x,t)}{h}$$

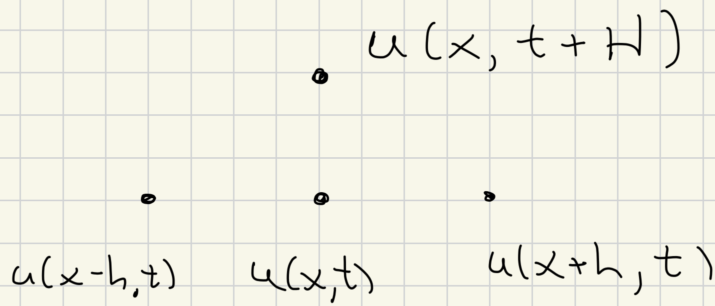


$u_{xx}(x,t)$ 2nd derivative

Recall $g: \mathbb{R} \to \mathbb{R}$, sufficiently differentiable

$$g''(x) = \lim_{h \to 0} \frac{g(x+h) + g(x-h) - 2g(x)}{h^2}$$

"three-point 2nd derivative approximation"



h, H small, eventually → 0

$$u_t(x,t) = c \; u_{xx}(x,t)$$

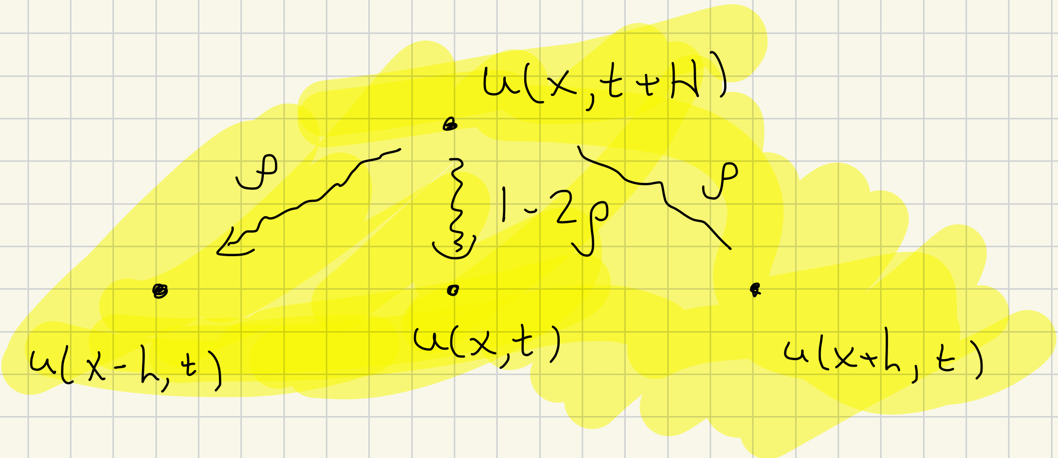
approx ↓

$$\frac{u(x,t+H) - u(x,t)}{H} \quad \Bigg\} \quad c \; \frac{\big(u(x+h,t) + u(x-h,t) - 2\,u(x,t)\big)}{h^2}$$

$$u(x,t+H) - u(x,t)$$

$$\approx \underbrace{\frac{cH}{h^2}}_{\rho}\Big(u(x+h,t) + u(x-h,t) - 2u(x,t)\Big)$$

$$u(x,t+H) = u(x,t) + \rho\Big(u(x+h,t) + u(x-h,t) - 2u(x,t)\Big)$$



e.g.

32 $\quad u(x,t) = 30 \quad$ 31 $\quad$ 28

Intuition:

$$u(x,0) = f(x) \quad \text{given}$$

$t=0$

$$u(0,t) = u(1,t) = 0$$

left endpoint   right endpoint

Example:

$$u(x,t) = \sin(\pi x)\, e^{-\pi^2 c t}$$

satisfies:

$$u_{xx} = \left(\sin(\pi x)\right)_{xx} e^{-\pi^2 c t}$$

$$\left(-\sin(\pi x)\pi^2\right)\left(\qquad \right)$$

$$u_t = \sin(\pi x)\left(e^{-\pi^2 c t}\right)_t$$

$$= \sin(\pi x)\left(-\pi^2 c \, e^{-\pi^2 c t}\right)$$

Similarly

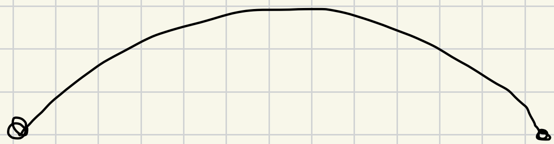
$$u(x,t) = u(m; x, t)$$

$1, 2, 3, \ldots$

$$= \left(\sin(\pi x m)\right) \left(\cdot\, e^{-\pi^2 c m t}\right)$$

also satisfies $u_t = c\, u_{xx}$

$$u(x,0) = \sin(\pi x)$$



$x=0$ $x=1$

$$u(x,t) = u(x,0)\,\underbrace{e^{-\pi^2 c t}}_{\text{decays exponentially in } t}$$

(1) $u_t = c\,u_{xx}$ has maximum principles

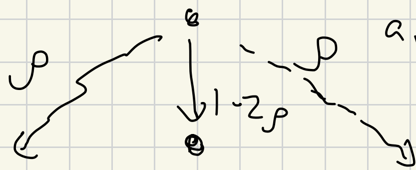
(2) " " " exact solutions

Numerical solution:

$u_t = c\, u_{xx}$, pick h = x-grid spacing

$u(x_i, t_j + H) = u(x_i, t_{j+1})$

approx



$u(x_i, t_j)$

$\vdots$

$t_2 = 2H$

$t_1 = H \quad u(0, H) \quad u(h, H) \quad - \; -$

$t_0 = 0 \quad u(0, 0) \quad u(h, 0) \quad u(2h, 0) \quad - \; - \; - \quad u(1, 0)$

$0 = x_0 \quad x_1 = h \quad x_2 = 2h \quad x_3 = 3h \quad \dots \quad x_m = 1$

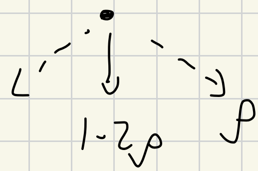
$\underbrace{\qquad\qquad}_{h}$

$mh = 1$

$h = \frac{1}{m}$

$$u(0, t_{j+1}) = 0$$

$$u(h, t_{j+1}) = (1-2\rho)\,u(h, t_j) + \rho\, u(h+h, t_j)$$



$$\vdots$$

$$u(x_i, t_{j+1}) = \rho\, u(x_{i-1}, t_j) + (1-2\rho)\, u(x_i, t_j) + \rho\, u(x_{i+1}, t_j)$$

$$\bar{U}(\ , t_j) = \begin{array}{c} 0 \\ \begin{bmatrix} u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{m-1}, t_j) \end{bmatrix} \\ 0 \end{array}$$

$$\bar{U}(\ , t_{j+1}) = (1-2\rho)\,\bar{U}(\ , t_j) + \rho \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & 0 \end{bmatrix} \bar{U}(\ , t_j)$$

$$= \bar{U}(\ , t_j)\,(1-2\rho) + \rho\, N_{rod,\, m-1}\, \bar{U}(t_j)$$

$$\bar{U}(\ , t_{j+1}) = \Big((1-2\rho)I + \rho N_{\mathrm{rod},m-1}\Big) U(\ t_j)$$

$$\begin{tikzcd}
 & x_i \arrow[dl] \arrow[d] \arrow[dr] & \\
x_{i-1} & x_i & x_{i+1}
\end{tikzcd}$$

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & & \ddots & & \\ & & \cdots & 0 & 1 & 0 \end{bmatrix}$$

$$U(\cdot\ t_j) = \Big((1-2\rho)I + \rho N\Big)^j U(\cdot\ t_0)$$

Problem:

$$\left\| (1-2\rho)I + \rho N \right\|_\infty \; ??$$

$$= \left\| \begin{pmatrix} 1-2\rho & \rho & 0 & 0 & \cdots \\ \rho & 1-2\rho & \rho & 0 & \cdots \\ 0 & \rho & 1-2\rho & \rho & 0 \cdots \\ & \vdots & & & \end{pmatrix} \right\|_\infty$$

$$= \begin{cases} 1 & \text{if } 0 \leq \rho \leq 1/2 \\ |1-2\rho| + 2\rho & \rho \geq 1/2 \end{cases}$$

$\Rightarrow$ if $\rho > 1/2$,

and you iterate

$$\Big(\underbrace{(1-2\rho)\, I + \rho N}_{\substack{\rho = 1 \\ \|\ \|_\infty = 3}}\Big)^j \xrightarrow[\text{bad}]{}$$

$$3^j \approx 2^{53}$$

$$3^j\, 2^{-53} \text{ gets } > 1 \ \ldots$$