CPSC 303, April 3, 2024

- More on

$$N_{rod,n} = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}$$

and the better behaved variant,

$N_{ring,n}$.

<u>Motivation 1:</u>

$$(4I + N_{rod,n})\vec{c} = 6 \overbrace{\begin{bmatrix} f[x_0,x_1,x_2] \\ \vdots \\ f[x_{n-2},x_{n-1},x_n] \end{bmatrix}}^{\Psi}$$

<u>Motivation 2:</u>

The heat equation $u_t = c\, u_{xx}$

Admin:

Questions about midterm grading?

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Subject: CPSC 303

Splines:

$$(4I + N_{rod,n}) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = 6 \overbrace{\begin{bmatrix} \text{triple} \\ \text{div} \\ \text{diff} \end{bmatrix}}^{\Psi}$$

c_i should be functions "mostly of

$f[x_{j-1}, x_j, x_{j+1}]$ for "j near i"

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \underbrace{(4I + N_{rod,n})^{-1}}_{???} \Psi 6$$

$$(4I+N)^{-1}$$

$$= \left[(4I)\left(I+(N/4)\right)\right]^{-1}$$

$$= \left(\frac{1}{4} I\right)\left(I - \left(N/4\right) + \left(N/4\right)^2 - \left(N/4\right)^3 + \cdots\right)$$

$$N = N_{\text{rod},n} = \begin{pmatrix} 0 & 1 & & \text{\Large 0} \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ \text{\Large 0} & & 1 & 0 \end{pmatrix}$$

$N^2 = ?$, $N^3 = ?$, ...

$$\|N\|_\infty = 2 \qquad (n \geq 3)$$

$$\| N/4 \|_\infty = \frac{1}{2}$$

$$I - \left(N/4\right) + \left(N/4\right)^2 - \left(N/4\right)^3 + \cdots$$

$$= \sum_{j=0}^{k} \left(-N/4\right)^k + \underbrace{\sum_{j=k+1}^{\infty} \left(-N/4\right)^k}$$

$$\| \quad \|_\infty \leq \left(1/2\right)^{k+1} + \left(1/2\right)^{k+2} + \cdots$$

$$= 1/2^k$$

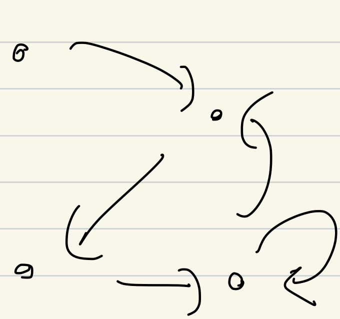
So infinite term

$$\left\| \sum_{j=k+1}^{\infty} \left(-N/4\right)^j \right\|_\infty \le \frac{1}{2^k}$$

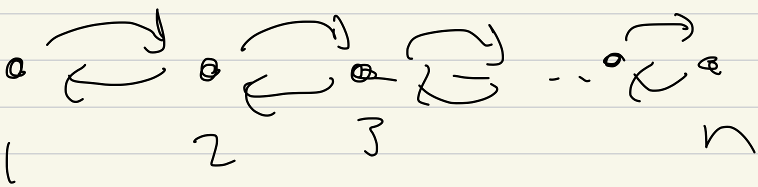
so this part — each sum of absolute values in each row — is $1/2^k$.

To understand $N_{\text{red},n} = \begin{bmatrix} 0 & 1 & & 0 \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{bmatrix}$

Directed graphs



Path length $n-1$



$$P_{n-1} = (V, E),$$

$$V = \{1, \ldots, n\}$$

$$E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}\}$$

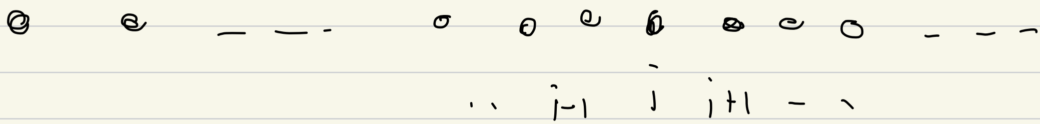
$A_{P_{n-1}}$ = adjacency matrix of P_{n-1}

$$= \begin{array}{c|c} & \begin{matrix} 1 & 2 & \cdots & & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ & & & & \ddots & \\ & & & & & \end{bmatrix} \end{array} = N_{rod,n}$$

$$(A_{P_{n-1}})_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

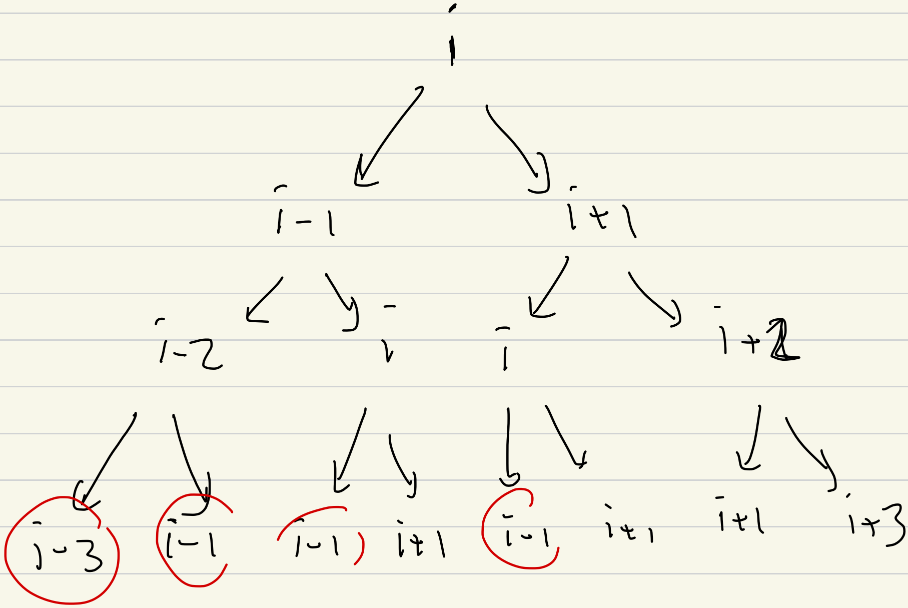
$A^k_{P_{n-1}}$ = # walks of length k in P_{n-1}

So, if $k = 3$

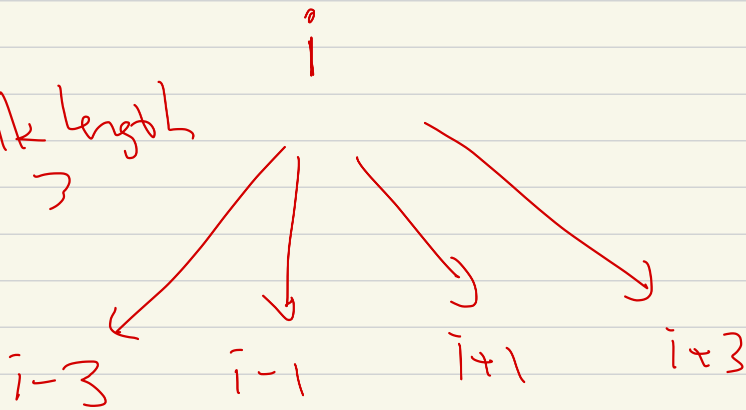


$$\left(A^3_{P_{n-1}}\right)_{ij}$$

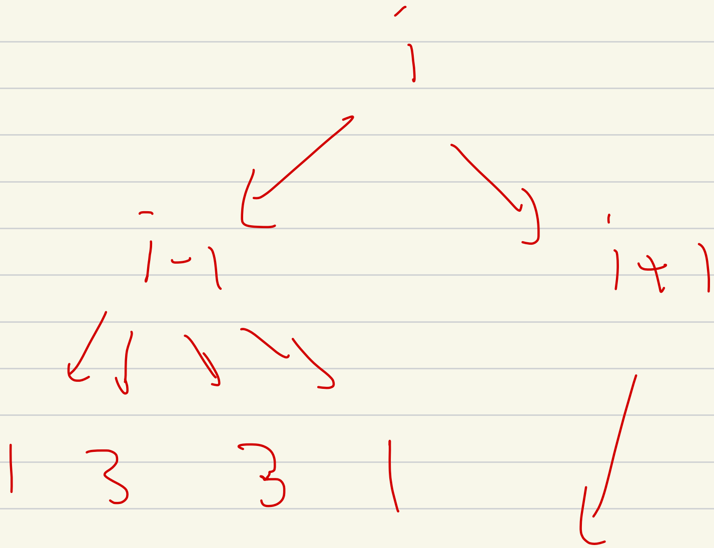
$=$ # walks in P_{n-1} from i to j



walk length 3



one walk	three walks	three walks	one walk
1	3	3	1



1 3 3 1

i-4 i-2 i i+2 i+4

$$\begin{array}{ccccccccc} 1 & & 3 & & 3 & & 1 & & \\ & & 1 & & 3 & & 3 & & 1 \\ \hline 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$N_{red,n} = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}}_{S_{n,1}} + \underbrace{\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}}_{S_{n,-1}}$$

"shift operators"

$$S_{n,1}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & & 0\text{'s} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0\text{'s} & & & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ 0 \end{bmatrix}$$

Cyclic Shift Matrix, n elements

$$= \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix}$$

$$C_{n,1} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix}$$

$$C_{n,1} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ x_1 \end{bmatrix}$$

$$C_{n,-1} = \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & 1 & \ddots & \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

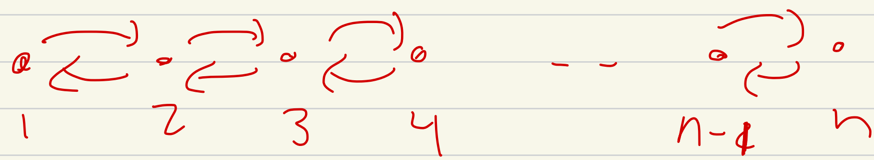
$$C_{n,-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$C_{n,-1} \cdot C_{n,1} = I$$

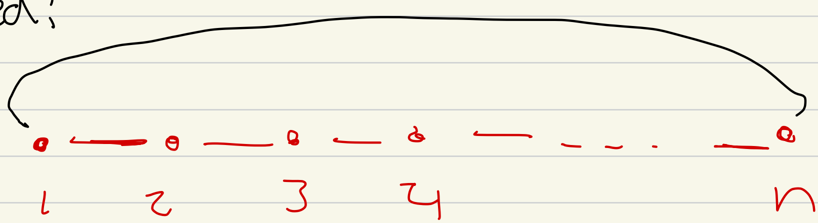
$$C_{n,1}^{-1} = C_{n,-1}, \quad C_{n,-1}^{-1} = C_{n,1}$$

$$C^{2}_{n,1}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \vdots \\ x_n \\ x_1 \\ x_2 \end{bmatrix}$$

Path length $n-1$



Added:



makes P_{n-1} + edge $\to$ Cycle Length$_n$

$$N_{\text{new variant}, n} = C_{n,1} + C_{n,-1}$$

$$= \begin{pmatrix} 0 & 1 & & & 1 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & 0 \end{pmatrix}$$

$$N^3 = \left(C_{n,1} + C_{n,-1}\right)^3$$

$$= \left(C_{n,1} + C_{n,1}^{-1}\right)^3$$

$$(x + x^{-1})^3$$

$$= x^{-3}(x^2+1)^3$$

$$= x^{-3}(1 + 3x^2 + 3x^4 + x^6)$$

$$= x^{-3} + 3x^{-1} + 3x + 3x^3$$

Since $C_{n,1}$ commutes with $C_{n,-1}$, and $C_{n,-1} = C_{n,1}^{-1}$

$$N_{new}^3 = (C_{n,1} + C_{n,1}^{-1})^3$$

$$= C_{n,1}^{-3} + 3C_{n,1}^{-1} + 3C_{n,1} + C_{n,1}^3$$

By contrast,

$S_{n,1}$, $S_{n,-1}$ are

"almost" but not "exactly"

inverses ...

$$N_{red,n} = \begin{bmatrix} 0 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{bmatrix}$$

entry-by-entry

$\leq$

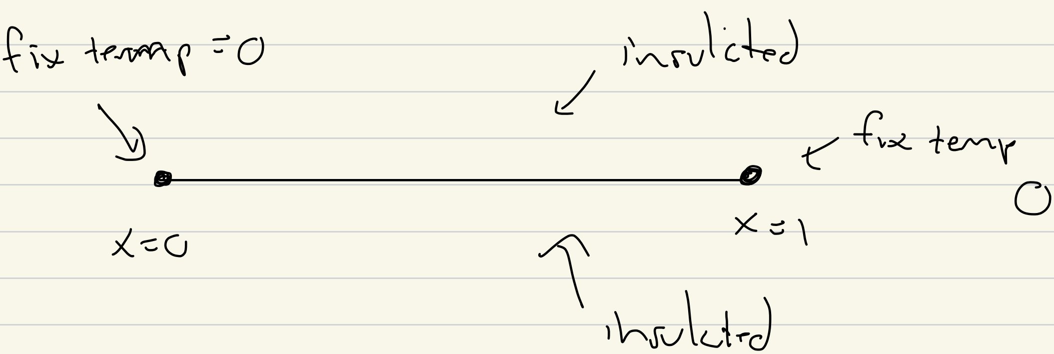
$$N_{new,n} = \begin{bmatrix} 0 & 1 & & 1 \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & 0 \end{bmatrix}$$

PDE: Heat equation:

$$u_t = c\, u_{xx}$$

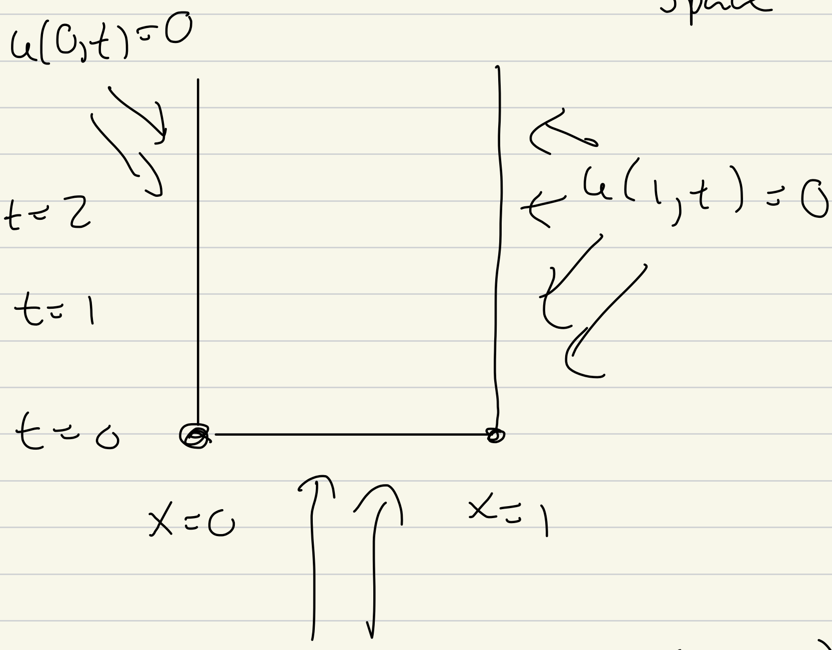
ODE's again ---

Think very thin rod, wire



Tcnp profile! $u = u(x, t)$

space   time



initial temp $u(x, 0) = f(x)$

$f(x)$ $t = 0$

. . . $\longrightarrow$ Nred,n arises . . .