CPSC 303, March 27, 2024

Today:

More on $N_{rod,n}$:

- Graphs, adjacency matrices.
- Adjacency matrices and $N_{rod,n}$.
- $N_{rod,n}$, $S_{n,1}$, $S_{n,-1}$ versus $N_{cycle,n}$, $C_{n,1}$, $C_{n,-1}$.
- $\left(I - \frac{1}{4} N_{rod,n}\right)^{-1}$ is "localized".

$$N_{red,n} := \begin{bmatrix} 0 & 1 & & & 0\text{'s} \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0\text{'s} & & & 1 & 0 \end{bmatrix}$$

Splines:

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

$$a_i, b_i, d_i \leftarrow f[x_i],\ f[x_{i+1}]$$

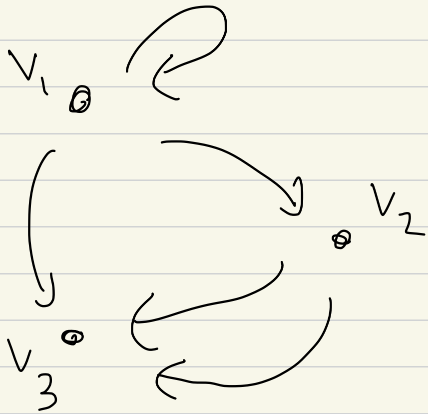
$$f[x_i, x_{i+1}]$$

$$\boxed{c_i,\ c_{i+1}}$$

x_0 x_1 x_2 … x_n

Graph theory

Directed graph:



Simple Directed Graph

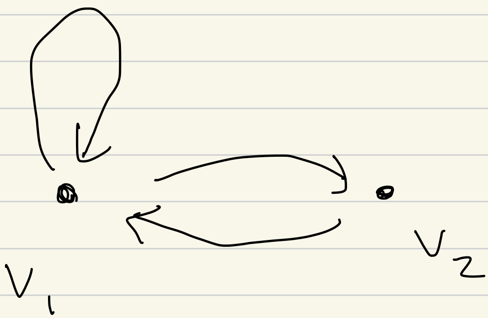
$G = (V, E)$

vertex set

edge set

$E = \{$ ordered pairs of $V \}$

Fibonacci directed graph:



$$V = \{ v_1, v_2 \}$$

$$E = \{ (v_1, v_1), (v_1, v_2), (v_2, v_1) \}$$

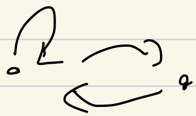
$$G = (V, E) = G_{\text{Fibonacci}}$$

$$A_{G_{\text{Fib}}} = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Adjacency matrix of a digraph (directed graph)

$G = (V, E)$ is the $|V| \times |V|$ matrix, square matrix whose entries are indexed on V, where

$$(A_G)_{(v,v')} = \begin{cases} 1 & \text{if } (v,v') \in E \\ 0 & \text{otherwise} \end{cases}$$

$A\Big($  $\Big) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$x_{n+2} = x_{n+1} + x_n$$

$$\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

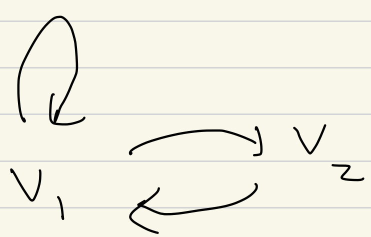
$$A^2_{G_{fib}} = \begin{array}{c} \\ v_1 \\ \\ v_2 \end{array} \begin{array}{c} \begin{array}{cc} v_1 & \qquad\qquad v_2 \end{array} \\ \begin{bmatrix} \text{\# of walks of length 2 from } v_1 \text{ to } v_1 & \text{- - -} \ v_1 \text{ to } v_2 \\ \text{- - -} \ v_2 \text{ to } v_1 & \text{- - -} \ v_2 \text{ to } v_2 \end{bmatrix} \end{array}$$

Prove, prove

$$A_G^m = \text{matrix that counts the number of walks of length } m \text{ in the graph}$$

$$\left(A_G^m\right)_{v_1, v_2} = \#\text{ walks of length } m \text{ from } v_1, v_2$$

Fibonacci Graph



Start v_1:

$$\left.\begin{array}{l} v_1 \to v_1 \to v_1 \\ v_1 \to v_2 \to v_1 \\ v_1 \to v_1 \to v_2 \\ v_2 \to v_1 \to v_1 \\ v_2 \to v_1 \to v_2 \end{array}\right\} \text{walk of length } 2$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$
\begin{array}{c|cc}
 & v_1 & v_2 \\
\hline
v_1 & 1 & 1 \\
v_2 & 1 & 0
\end{array}
\qquad
\begin{array}{c|cc}
 & v_1 & v_2 \\
\hline
v_1 & 1 & 1 \\
v_2 & 1 & 0
\end{array}
$$

$$
\begin{array}{c|cc}
 & v_1 & v_2 \\
\hline
v_1 & 1 & 1 \\
v_2 & ? & ?
\end{array}
$$

1 1 means

$v_1 \to v_1$

$v_1 \to v_2$

$$
\begin{array}{c|cc}
 & v_1 & v_2 \\
\hline
v_1 & 1 & 1 \\
v_2 & 1 & 0
\end{array}
$$

$\dashrightarrow v_1 \to v_1$

$ \to v_2$

$\dashrightarrow v_2 \to v_1$

$ \cancel{\to v_2}$

$$A^3_{Fib.G} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

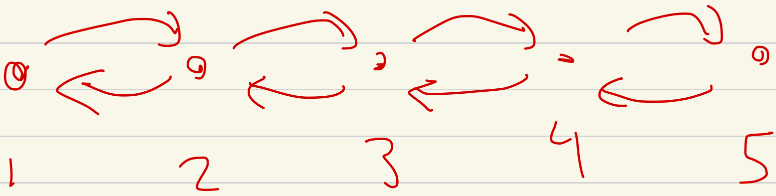
$$= \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \ldots$$

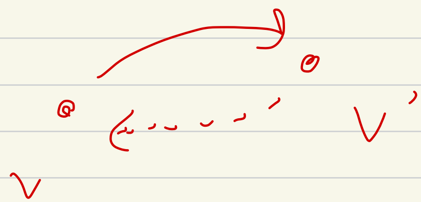
Path Digraph "Directed Path Length 5"

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$$

Often "path"



often!



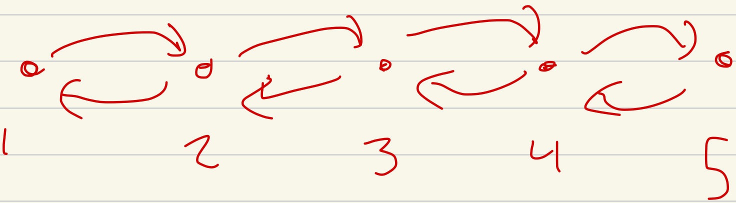
if $(v, v') \in E$, also $(v', v) \in E$

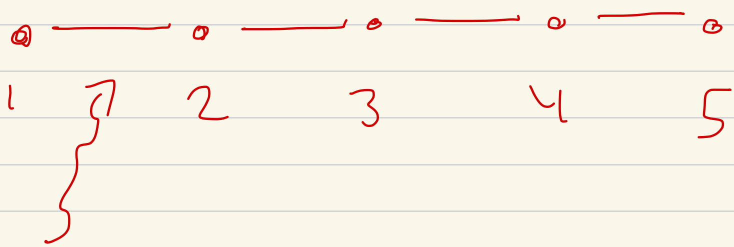
Graph: Just a directed graph

$G = (V, E)$ where

$(v, v') \in E$ then $(v', v) \in E$

(Undirected) Path of Length 5



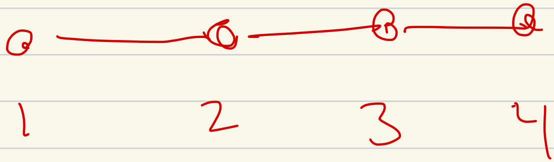


undirected edge

Simple Graph $G = (V, E)$

E contains unordered pairs $\{v_1, v_2\}$

$$A_{\text{undirected path length } 4} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & & 0\text{s} \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ 0\text{s} & & 1 & 0 \end{pmatrix} \end{array}$$

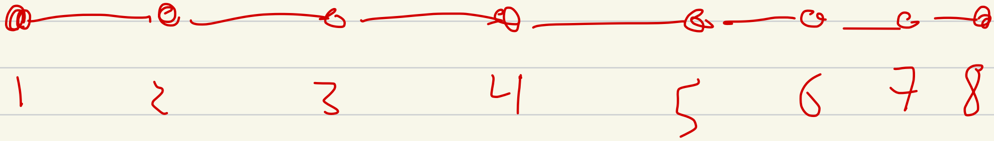


$\uparrow$

U

$N_{red,4}$

$$A^2 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$



$A^2_{8\text{-path}}$ = walks of length 2 in 8-path

$$= \begin{pmatrix} 1 & 0 & 1 & & & \text{0's} & & \\ 0 & 2 & 0 & 1 & & & & \\ 1 & 0 & 2 & \ddots & \ddots & & & \\ & 1 & \ddots & 2 & \ddots & \ddots & & \\ & & \ddots & \ddots & 2 & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & \ddots & 1 \\ & & & & \ddots & \ddots & 2 & 0 \\ & \text{0's} & & & & 1 & 0 & 1 \end{pmatrix}$$

$A^3_{8\text{-path}}$

Conclusions:

$$N_{rod,n} = A_{\substack{\text{path} \\ \text{length} \\ n}}$$

then:

$$\left(N_{rod,n}^{\,k}\right)_{i,j} = \text{must be } 0 \text{ if } |i-j|>k$$