CPSC 303, March 25, 2024

Last time: for natural splines

a_i, b_i, d_i = functions of

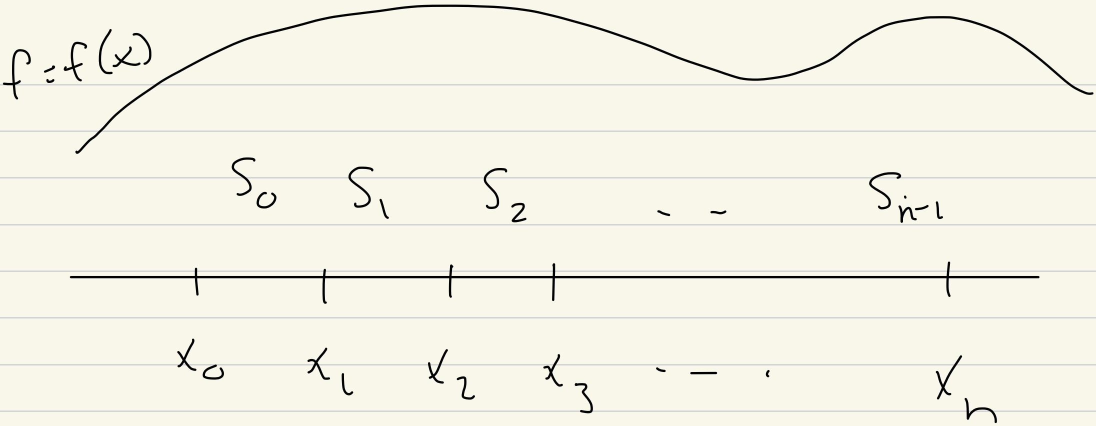
$f(x_i)$, $f[x_i, x_{i+1}]$, c_i, c_{i+1}

and

$$\underbrace{\left(2 + \frac{1}{2} N_{\mathrm{rod},n}\right)}_{\substack{\text{equally spaced} \\ x_0, \ldots, x_n}} \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = 3 \begin{bmatrix} f[x_0, x_1, x_2] \\ \vdots \\ f[x_{n-2}, x_{n-1}, x_n] \end{bmatrix}$$

Today: (1) So what?

(2) $N_{\mathrm{rod},n}$, Laplacian, ...



$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

$$\vec{a} = \begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix}, \quad \vec{b}, \vec{d} \quad \text{depend}$$

"locally on $\vec{c}$"

When $h = x_1 - x_0 = x_2 - x_1 = \cdots$

the system for $(c_0 = 0)$

$$\left(\underline{I} + \frac{1}{4} N_{rod,n}\right)\begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \frac{3}{2}\begin{bmatrix} f[x_0, x_1, x_2] \\ f[x_1, x_2, x_3] \\ \vdots \\ \vdots \end{bmatrix}$$

$$N_{rod,n} = \begin{bmatrix} 0 & 1 & & 0' \\ 1 & 0 & 1 & \\ & 1 & 0 & \ddots \\ & & 0 & \ddots & \ddots \\ 0' & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}$$

$$N_{rod,n}\begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} y_2 \\ y_1 + y_3 \\ y_2 + y_4 \\ \vdots \end{bmatrix} \leftarrow$$

j^{th} component depends only on y_{i-1}, y_{i+1}

$$\left(I + \frac{1}{4} N_{rod,n}\right) \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{n-1} \end{bmatrix}$$

(essentially Ψ is [A&G])

$$\| N_{rod,n} \|_\infty = 2 \quad \left(\text{maybe } 1 \text{ when } n=2\right)$$

$$\begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \left(I + \frac{1}{4} N_{rod,n}\right)^{-1} \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{n-1} \end{bmatrix}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots, \quad |x| < 1$$

$(1-y)^{-1} = 1 + y + y^2 + \dots \qquad |y| < 1$

If $\|A\|_\infty < 1$ (really $\|A\|_p < 1$ any p)

claim:

$$(I-A)^{-1} = I + A + A^2 + A^3 + \dots$$

Why?

Does each entry of $I + A + A^2 + \dots$

converge

Example: $A = \left[\, 3/4 \,\right]$

$$(I-A)^{-1} = \left([1] - \left[\tfrac{3}{4}\right] \right)^{-1} = \left[\tfrac{1}{4}\right]^{-1}$$

$$I + A + A^2 = [1] + [3/4] + \dots$$

$$\left(1-\frac{3}{4}\right)^{-1} = 1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \dots$$

Example: $A = \begin{bmatrix} 3/4 & 0 \\ 0 & 3/4 \end{bmatrix}$ ...

$$I + A + A^2 + \dots = \begin{bmatrix} 1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\dots & 0 \\ 0 & 1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\dots \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 3/4 & 0 \\ 0 & -1/3 \end{bmatrix}$$

or

$$A = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad |d_1|, |d_2| < 1$$

Example!

$$A = S \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} S^{-1}$$

$|d_1|, |d_2| < 1$, then

$$I + A + A^2 + A^3 + \ldots$$

$$= S \left(I + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} + \begin{bmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{bmatrix} + \ldots \right) S^{-1}$$

$$(I - A)^{-1} = \left(I - S \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} S^{-1} \right)^{-1}$$

$$= \left(S I S^{-1} - S \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} S^{-1} \right)^{-1}$$

$$\left(S \left(I - \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \right) S^{-1} \right)^{-1}$$

$$(S^{-1})^{-1} \left(I - \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \right)^{-1} S^{-1}$$

$$= S \underbrace{\left(I - \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \right)^{-1}}_{I + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} + \cdots + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}^m + \underbrace{\cdots}_{\text{leftover}}} S^{-1}$$

leftover! $\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}^{m+1} + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}^{m+2} + \cdots$

$$\begin{bmatrix} d_1^{m+1}(1+d_1+d_1^2+\cdots) & 0 \\ 0 & \text{similarly} \end{bmatrix}$$

each entry

$$\leq \max_{d=d_1,d_2} |d|^{m+1}(1+|d|+\cdots)$$

So... (intuitively)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} + \begin{bmatrix} & \\ & \end{bmatrix}^2 + \cdots$$

Remark:

$$\|AB\vec{v}\|_\infty \leq \|A(B\vec{v})\|_\infty$$

$$\leq \|A\|_\infty \|B\vec{v}\|_\infty$$

$$\leq \|A\|_\infty \|B\|_\infty \|\vec{v}\|_\infty$$

So (matrix norm condition)

$$\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p$$

$$\|A^2\|_\infty = \|AA\|_\infty$$

$$\leq \|A\|_\infty \|A\|_\infty = \left(\|A\|_\infty\right)^2$$

$$\|A^3\|_\infty \leq \left(\|A\|_\infty\right)^3$$

$$\vdots$$

Rem:

$$N_{rod,n} = \mathrm{Shift}_{n,1} + \mathrm{Shift}_{n,-1}$$

$$= \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots & 1 \\ 0 & & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & \ddots \\ 0 & & & 1 & 0 \end{bmatrix}$$

$$\mathrm{Shift}_{n,1} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ & & & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \\ 0 \end{bmatrix}$$

$$\mathrm{Shift}_{2,1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathrm{Shift}_{2,1}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

E.g., $\|\mathrm{Shift}_{n,1}\|_\infty = 1 \quad (n \geq 2)$

$$\mathrm{Shift}_{n,1}^{n} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & 0 \end{bmatrix}$$

$$\left\| \mathrm{Shift}_{n,1}^{n} \right\|_\infty = 0$$

$$\leq \left\| \mathrm{Shift}_{n,1} \right\|_\infty^{n}$$

$$= 1^{n} = 1$$

$$\left(I + \frac{1}{4} N_{\mathrm{red},n-1} \right)^{-1}$$

$$= I - \frac{1}{4} N + \frac{1}{16} N^2 - \frac{1}{64} N^3 + \frac{1}{4^4} N^4 + \cdots$$

$$\vec{c} = \left(I + \frac{1}{4}N\right)^{-1} \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{n-1} \end{bmatrix} \leftarrow f[x_{i-1}, x_i, x_{i+1}]$$

div diff

$$= \left(I - \frac{1}{4}N + \frac{1}{16}N^2 - \cdots\right) \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{n-1} \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$N^2 = \begin{bmatrix} \\ \\ \end{bmatrix} \quad ?$$

$$N = \begin{pmatrix} \ddots & \ddots & & 0's \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0's & & \ddots & \ddots \end{pmatrix}$$

"tri-diagonal"

A is tri-diagonal

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ a_{31} & & & \\ \vdots & & & \\ a_{n1} & & & \end{bmatrix}$$

If

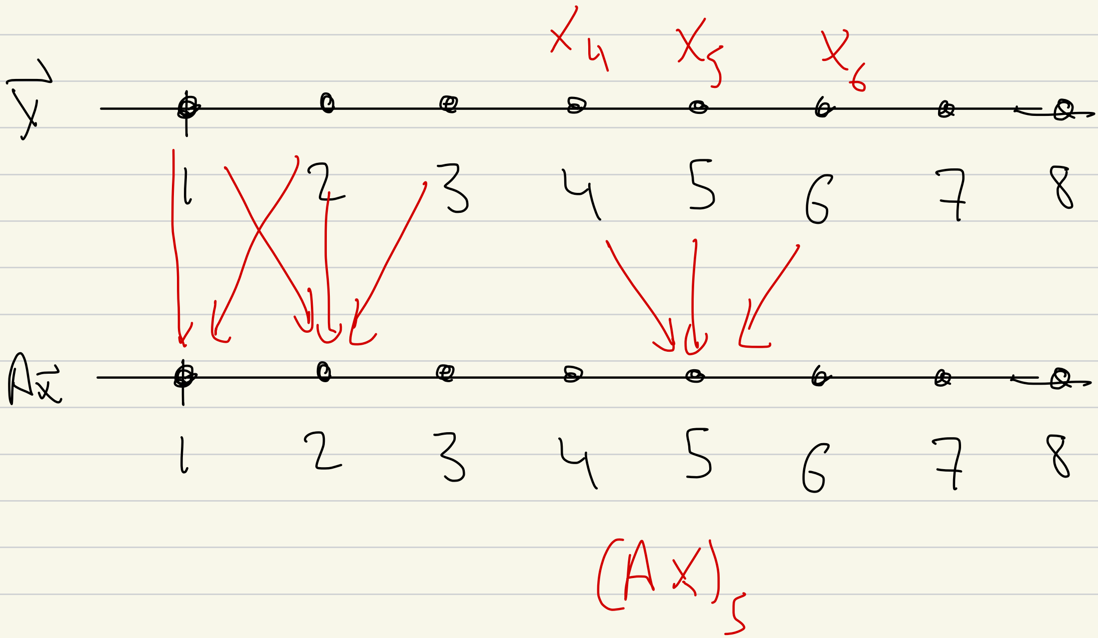
$a_{ij} \neq 0$

then

$|i-j| \leq 1$

$$\begin{bmatrix} \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}$$

$$\begin{bmatrix} \ddots & \ddots & & 0\text{'s} \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0\text{'s} & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \ddots & \ddots & & 0\text{'s} \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0\text{'s} & & \ddots & \ddots \end{bmatrix}$$



$$\begin{bmatrix} & & \\ & \text{tridiag} & \\ & & \end{bmatrix} \begin{pmatrix} x_1 \\ | \\ | \\ x_n \end{pmatrix} = \begin{pmatrix} \text{depends on } x_1, x_2 \\ \text{'' \quad ''} \; x_1, x_2, x_3 \\ x_2, x_3, x_4 \\ \vdots \\ \vdots \end{pmatrix}$$