CPSC 303, March 22, 2024

Today: Remarks on HW 8 +

Thm: Assume $v(x)$ is a cubic spline with abscissae $A = x_0 < \dots < x_n = B$,

$v(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$

s.t. $v \in C^2[A,B]$ and

$v''(A) = v''(B) = 0$. Say $v(x_i) = f(x_i)$.

Then (1) $c_0 = 0$

(2) a_i, b_i, d_i are functions of $c_1, \dots, c_{n-1}$ and the divided differences of f.

$$\left[\begin{array}{l} a_i = f(x_i) \quad \checkmark \\[2ex] d_i = \dfrac{c_{i+1} - c_i}{3h_i}, \qquad h_i = x_{i+1} - x_i \\[2ex] b_i = f[x_i, x_{i+1}] - \dfrac{h_i}{3}(2c_i + c_{i+1}) \end{array}\right]$$

(3)

$$\boxed{\begin{array}{l} h_{i-1} c_{i-1} + 2(h_{i-1} + h_i) c_i \\ \quad + h_i c_{i+1} = \\ \quad 3\left(f[x_i, x_{i+1}] - f[x_{i-1}, x_i] \right) \end{array}}$$

or equivalently (3'') surprize!...

Rem:

Homework 8 and Monday's class

$$\int_{x=x_0}^{x=x_1} \text{something}(x)\, g(x)\, dx = 0$$

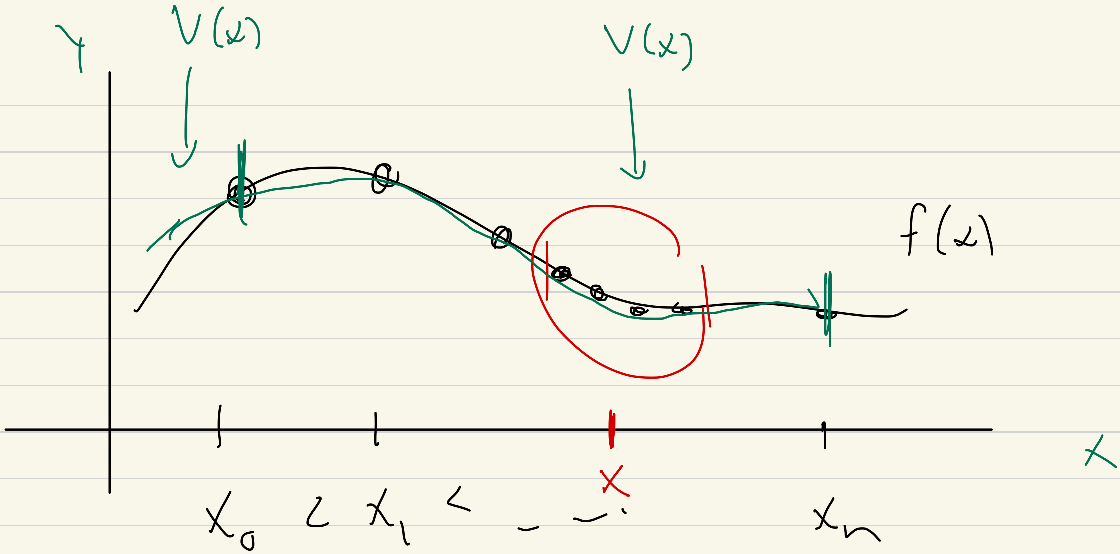
for all $g(x)$, $g : [x_0, x_1] \to \mathbb{R}$

s.t.

$g = 0$ near x_0

$g = 0$ near x_1

$$\begin{cases} g \in C^{20}[x_0, x_1] \\ g \in C^{\infty}[x_0, x_1] \\ \vdots \end{cases}$$



Want $v(x)$ to be determined "locally" by values of f near x

surprize

+ HW 8 problem 4

$$\begin{bmatrix} 0 & 1 & & 0's \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \\ 0's & & 1 & 0 \end{bmatrix}$$

Monday!

Say $\text{Energy}_1(u) = \displaystyle\int_{x=A}^{x=B} [u'(x)]^2 \, dx$

Say

$$\boxed{\text{Energy}_1(v) \leq \text{Energy}(v + \varepsilon g)} \; *$$

for some g

$$\underbrace{\alpha}_{\text{Energy}_1(v)} + \beta \varepsilon + \underbrace{\gamma}_{\text{Energy}(g)} \varepsilon^2$$

$\beta \Rightarrow$ cross term

$$= 2 \int_{x=A}^{B} v'(x)\, g'(x)\, dx$$

$$= 2 \left(v'(x)\, g(x) \Big|_{A}^{B} - \int_{A}^{B} v''(x)\, g(x)\, dx \right)$$

$g(x) = 0$ at $x = A, B$

So $(*) \Rightarrow$ $\int_{A}^{B} v''(x)\, g(x)\, dx = 0$

for any g --

$\to v''(x)$ HW8: $v''''(x)$

$$\int_A^B \text{something}(x) \cdot g(x)\, dx = 0$$

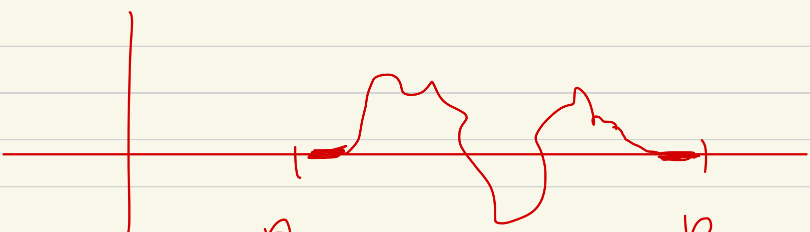
for all g --- nice

If

something: $[A,B] \to \mathbb{R}$

continuous, then

$g = 0$ near $x=A$, $x=B$

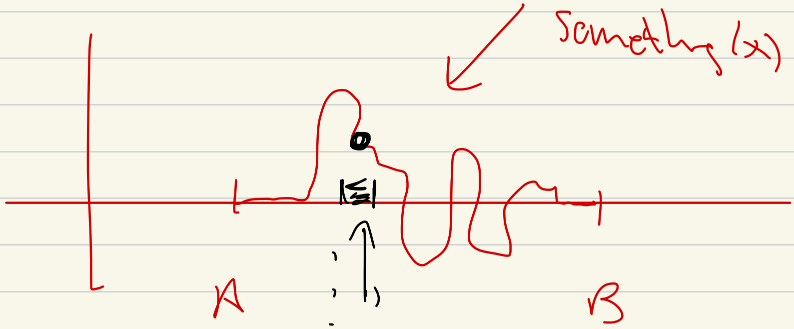


A B

$\Rightarrow$ something$(x) = 0$ on A, B

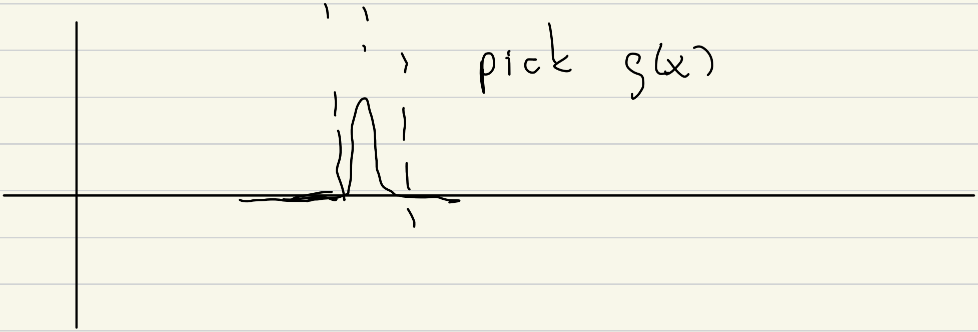
$\uparrow \quad \uparrow$

$x_0 \quad x_1$

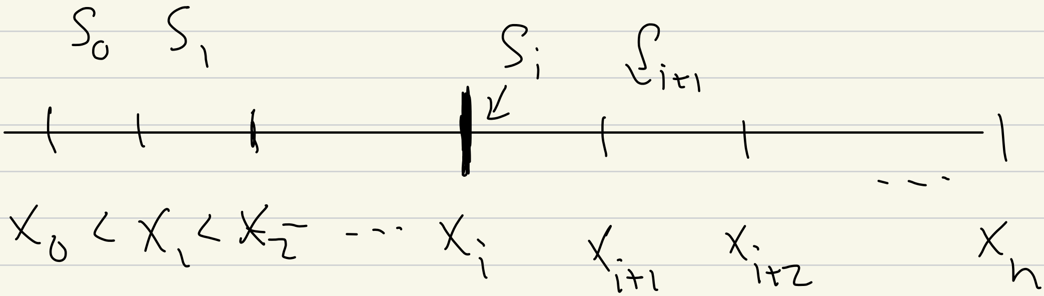


something$(x) \neq 0$

pick $g(x)$



$\Rightarrow$ contradiction



$V(x)$ s.t.

$$V(x) = S_i(x) \qquad x_i \le x \le x_{i+1}$$

$$\|$$

$$a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$$

Assume: $f : \mathbb{R} \to \mathbb{R}$

$$f(x_i) = \underbrace{V(x_i)}_{S_i(x_i) = a_i} \quad \text{for all } i = 0, \dots, n$$

$$S_i'(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2$$

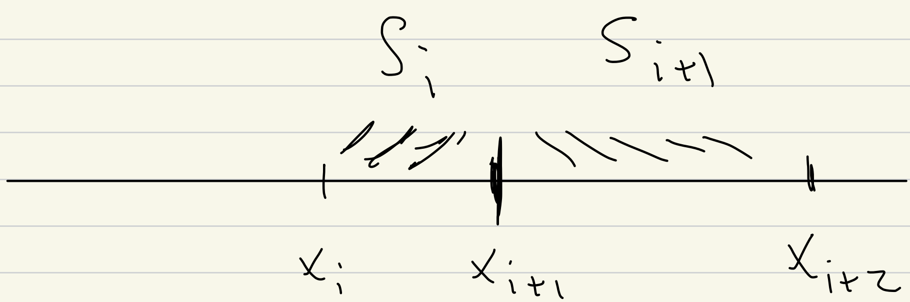
$$S_i''(x) = 2c_i + 3d_i\, 2(x - x_i)$$

$$= 2c_i + 6d_i(x - x_i)$$

$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}) = 2c_{i+1}$$

$$\parallel$$

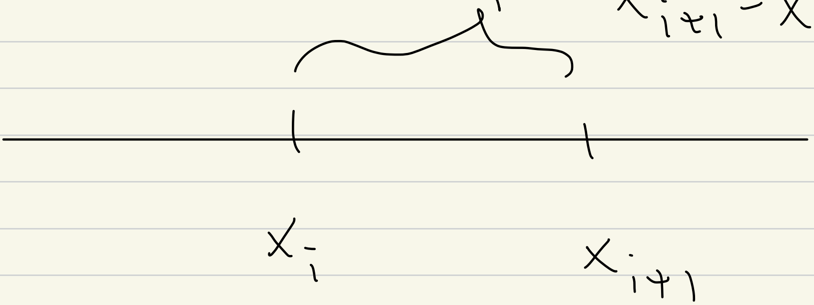
$$2c_i + 6d_i(x_{i+1} - x_i)$$



$$2c_i + 6d_i(x_{i+1} - x_i) = 2c_{i+1}$$

$$d_i = \frac{2c_{i+1} - 2c_i}{6(x_{i+1} - x_i)}$$

$$h_i = x_{i+1} - x_i$$

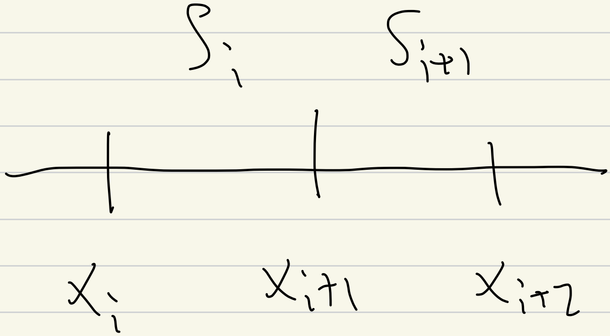


2nd derivative

$$\boxed{d_i = \frac{1}{3}\,\frac{c_{i+1} - c_i}{h_i}}$$

3rd der

b_i's can be written as c_i's



used values of S_i and a_i

" " " S_i'' and

d's $\leftarrow$ c's

$$S_i'(x_{i+1}) = S_{i+1}'(x_i)$$

Bottom line

b_i's —— c_i's , $f[x_i, x_{i+1}]$

$\vdots \qquad = \dfrac{f(x_{i+1}) - f(x_i)}{h_i}$

$\rightsquigarrow$ 

$$\longrightarrow h_{i-1} c_{i-1} + 2(h_{i-1} + h_i) c_i + h_i c_{i+1} = 3\left(f[x_i, x_{i+1}] - f[x_{i-1}, x_i] \right)$$

$i = 1, \dots, n-1$

Let's simplify ... let's write it

more intuitively ...

divide by

$$h_{i-1} + h_i = (x_i - x_{i-1}) + (x_{i+1} - x_i)$$

$$= x_{i+1} - x_{i-1}$$

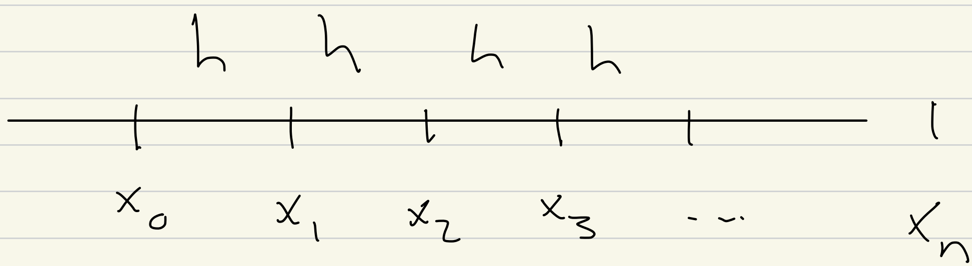
$$\frac{h_{i-1}}{h_{i-1}+h_i} c_{i-1} + 2c_i + \frac{h_i}{h_{i-1}+h_i} c_{i+1}$$

$$= 3 \, \frac{f[x_i, x_{i+1}] - f[x_{i-1}, x_i]}{x_{i+1} - x_{i-1}}$$

$$= 3 \underbrace{f[x_{i-1}, x_i, x_{i+1}]}$$

we know $= \dfrac{f''(\xi)}{2}$

$$x_{i-1} < \xi < x_{i+1}$$



$$h = h_0 = h_1 = \dots = h_{n-1}$$

$$\frac{h}{2h} c_{i-1} + 2c_i + \frac{h}{2h} c_{i+1}$$

$$= 3 \underbrace{f[x_{i-1}, x_i, x_{i+1}]}$$

$$\frac{1}{2} c_{i-1} + 2c_i + \frac{1}{2} c_{i+1}$$

$$\frac{1}{4} c_{i-1} + c_i + \frac{1}{4} c_{i+1} = \frac{3}{2} \text{ dv diff}$$

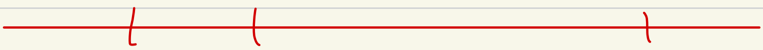
$$\begin{bmatrix} 1 & \frac{1}{4} & & & & \\ \frac{1}{4} & 1 & \frac{1}{4} & & & \\ & \frac{1}{4} & 1 & \frac{1}{4} & & \\ & & \frac{1}{4} & 1 & \frac{1}{4} & \\ & & & & \ddots & \\ & & & & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \frac{3}{2} \begin{bmatrix} \text{sliding} \\ \text{diffdiv} \\ \text{diff} \\ \\ \end{bmatrix}$$

0 $\qquad$ 0

Rem!

$$v''(A) = 0$$

$$v''(A) = s_d''(A) = 2c_0 \Rightarrow c_0 = 0$$



$A = x_0 \quad x_1 \quad \cdots$