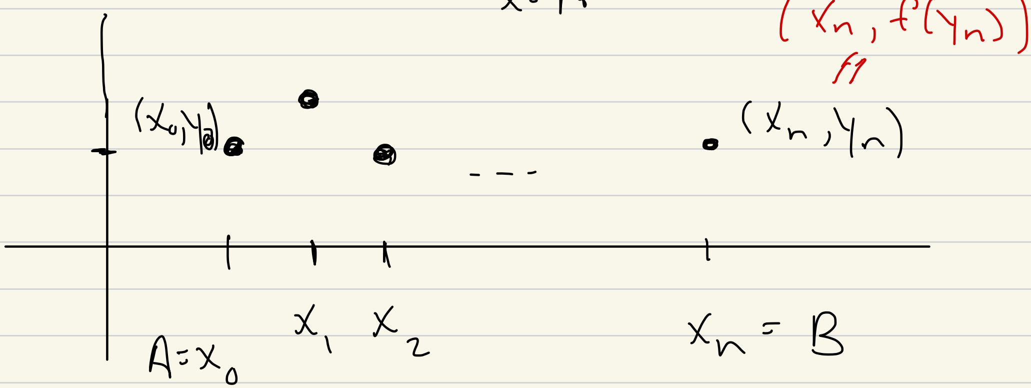
CPSC 303, March 20, 2024

TODAY

- Cubic splines minimize

$$\text{Energy}_2(u) = \int_{x=A}^{x=B} \left(u''(x)\right)^2 \, dx$$



over

$$\mathcal{U} = \left\{ u \in C^2[A,B] \;\middle|\; u(x_i) = y_i, \; i = 0, \ldots, n \right\}$$

- Boundary conditions:

$\mathcal{U}^{\text{free}} = \mathcal{U}$, $\mathcal{U}^{\text{clamped}}_{\alpha,\beta}$, etc.

Ch 10 :

(x_i, y_i) Section 2,3

Section 4

(x_i, y_i) Divided Diff

$\Downarrow$

$(x_i, f(x_i))$

Same interpolation, but §10.4

we measured

Diff in Interpolation as

$$|x-x_0|\,|x-x_1| \cdots |x-x_n| \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

viewing as fitting a function

Last time: $u : [A,B] \to \mathbb{R}$

$$\text{Energy}_1(u) = \int_{x=A}^{x=B} \left(u'(x)\right)^2 \, dx$$

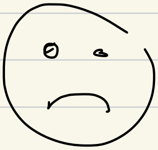
and fix $(x_0, y_0), \dots, (x_n, y_n)$

$$\mathcal{U}^1_{A,B} = \left\{ u \in C^1[A,B], \; u(x_i) = y_i \quad i = 0, \dots, n \right\}$$

we essentially proved:

there is no $v \in \mathcal{U}^1_{A,B}$ s.t.

$$\text{Energy}_1(v) \leq \text{Energy}_1(u)$$

 ...

for all $u \in \mathcal{U}^1_{A,B}$

Last time: if $u : (x_0, x_1] \to \mathbb{R}$

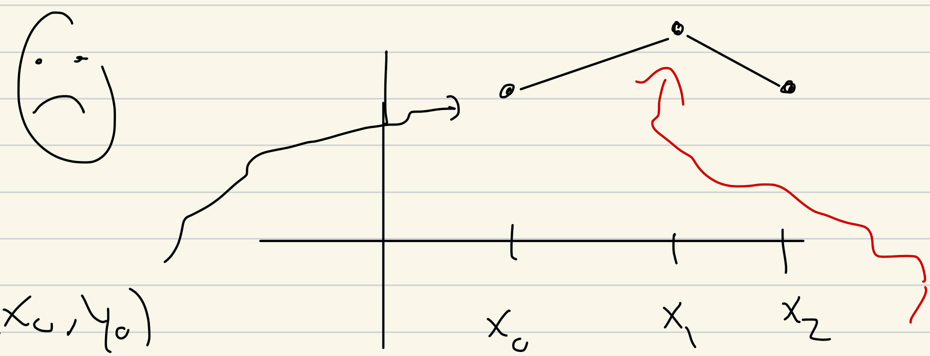
continuously differentiable, $u \in C^1[x_0, x_1]$

and $u(x_0)$, $u(x_1)$ given, and

u minimizes $\text{Energy}_1(u)$ or

$$\text{Length}(u) = \int_{x=x_0}^{x=x_1} \sqrt{1 + (u'(x))^2}\, dx$$

then $u''(x) = 0$ for all $x \in (x_0, x_1)$



(x_0, y_0)
$= (x_0, u(x_0))$

$u'(x_1)$
doesn't usually exist

Beyond this course...

$$\int_{x=A}^{x=B} \left(u'(x)\right)^2 dx$$

There is truly a space of functions

$W^{1,2}[A,B]$ "Sobolev space"

of the functions where

$\text{Energy}_{1,A,B}$, $\text{Length}_{A,B}$ ...

makes sense

$$W^{k,p}[A,B] = \left\{ u \text{ s.t. weakly } k\text{-times differentiable with } u^{(k)} \in L^p[A,B] \right\}$$

For now, forget this...

<u>Def</u>: Let $(x_0, y_0), \ldots (x_n, y_n)$ be fixed $A = x_0 < x_1 < \ldots < x_n = B$

Let

$$\mathcal{U} = \left\{ u \in C^2[A,B] \text{ s.t. } u(x_i) = y_i \text{ for } i = 0, \ldots, n \right\}$$

$$\text{Energy}_2(u) = \int_{x=A}^{x=B} \left(u''(x)\right)^2 dx$$

Then:

① there is a $v \in \mathcal{U}$ on which $\text{Energy}_2 : \mathcal{U} \to \mathbb{R}$ is minimized

(2) for this v:

$$v''''(x) = 0 \quad \text{for } x_i < x < x_{i+1}$$

(3) Also:

$$v''(A) = 0 \quad , \quad v''(B) = 0 .$$

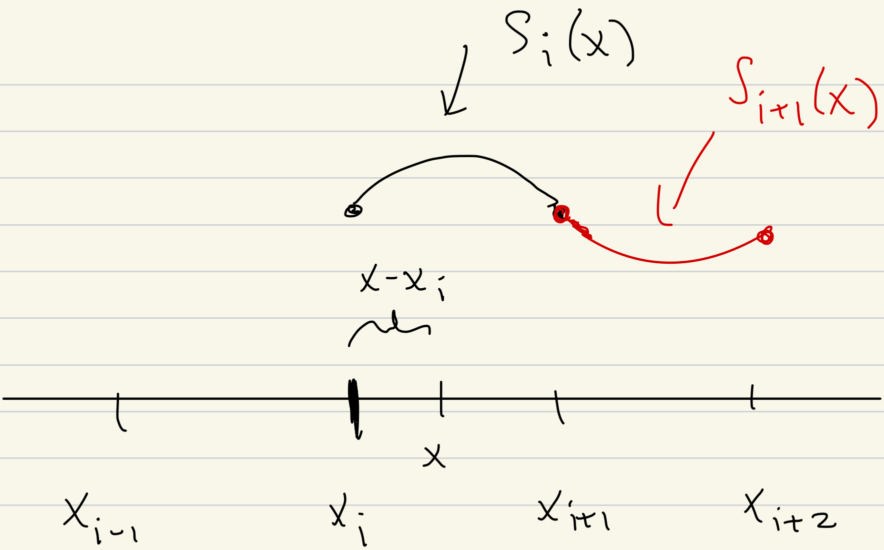
In other words

$$v''''(x) \qquad x_i < x < x_{i+1}$$

means v piecewise cubic poly,

i.e.

$$v(x) = \qquad\qquad x_i < x < x_{i+1}$$

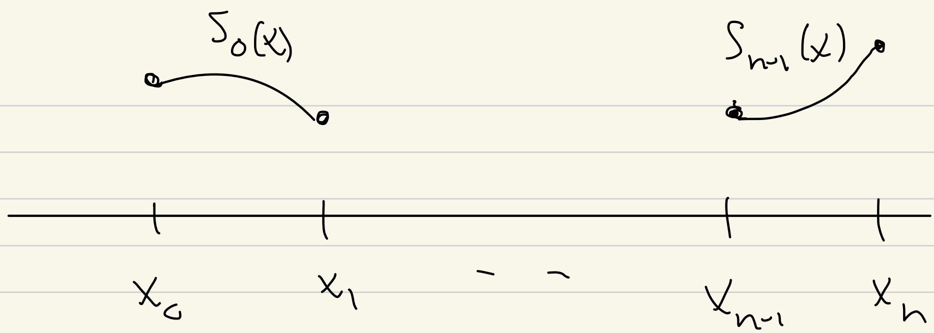


$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d(x - x_i)^3$$

and

$$v(x) = S_i(x) \quad \text{for } x_i \le x \le x_{i+1}$$

$$a_i, b_i, c_i, d_i \in \mathbb{R} \quad \text{for } i = 0, \dots, n-1$$



Let's believe the theorem ...

Linear alg w/o lin alg:

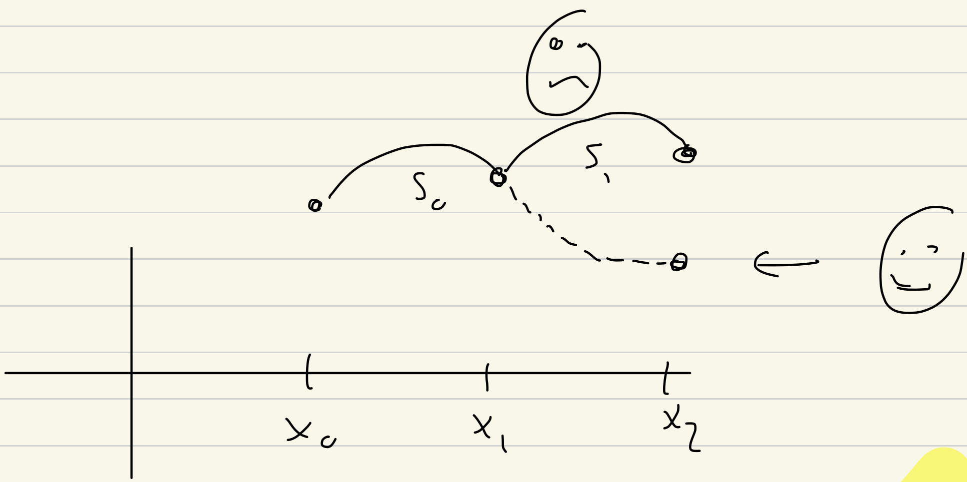
$$\underbrace{\{a_i, b_i, c_i, d_i\}}_{4}, \quad \underbrace{i = 0, \ldots, n-1}_{\substack{n \text{ patches,} \\ S_0, \ldots, S_{n-1} \\ \text{locally cubic}}}$$

$= 4n$ variables (parameters)

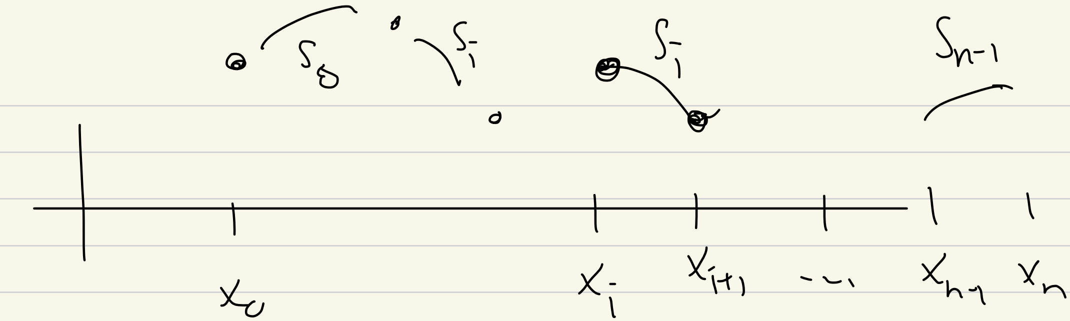
$$V(x) = \begin{cases} S_0(x) & x_0 \le x \le x_1 \\ S_1(x) & x_1 \le x \le x_2 \\ \vdots & \vdots \end{cases}$$

How many equation do we get s.t.

$$S_0(x_1) = y_1 = S_1(x_1)$$



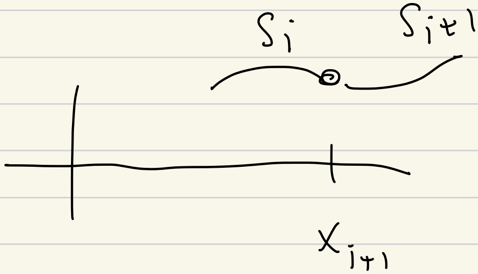
$$S_0'(x_1) = S_1'(x_1), \quad S_0''(x_1) = S_1''(x_1)$$



Values of S_i:

$$\left.\begin{aligned} S_i(x_i) &= y_i \\ S_i(x_{i+1}) &= y_{i+1} \end{aligned}\right\} \text{2 conditions}$$

$i = 0, \dots, n-1$ $\quad$ $2n$ conditions



S_i' match S_{i+1}'

S_i'' match S_{i+1}''

2 per $\overset{S_0 \quad S_1}{x_1}$ ... $\overset{S_{n-2} \quad S_{n-1}}{x_{n-1}}$

$2(n-1)$

Total # linear eqn:

$$2n + 2(n-1) = 4n-2$$

Thm: Energy_2 minimized when

$V''(x_0) = 0$, $V''(x_n) = 0$

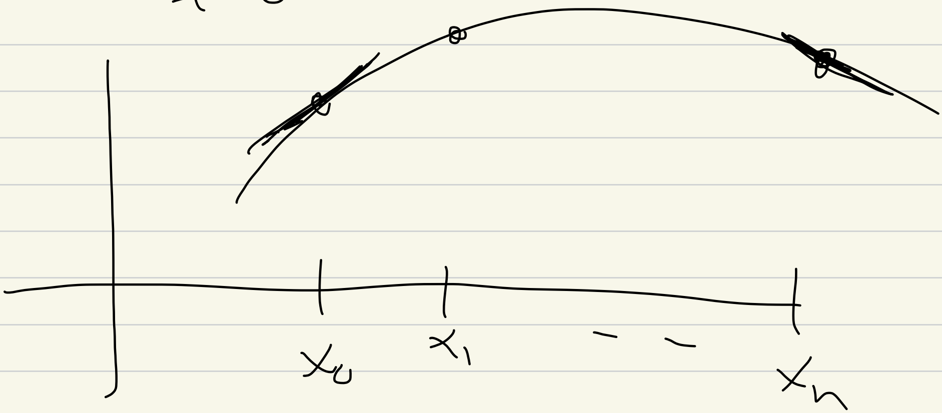
$(S''(x_0))$ $(S''_{n-1}(x_n))$

2 conditions

$4n$ variables, $4n-2$ conditions

on being in $C^2[A,B]$, + 2 conditions

f unknown



say $f(x_0), \ldots, f(x_n)$

say you know

$f'(x_0), \quad f'(x_n)$

Thm: If you fix $v'(x_0) = y_0'$, $v'(x_n) = y_n'$, you also look at

$$\mathcal{U}_{2,A,B,y_0',y_n'} = \left\{ u \in C^2[A,B] \;\middle|\; \begin{array}{l} u(x_i) = y_i \\ i = 0, \dots, n \\ u'(x_0) = y_0' \\ u'(x_n) = y_n' \end{array} \right\}$$

then $\exists$ unique $v \in \mathcal{U}_{2,A,B,y_0',y_n'}$ ...

that minimizes Energy_2,

$4n$ variables a_i, b_i, c_i, d_i $\quad i = 0, \dots, n-1$

$4n$ equations,

unique solution