CPSC 303, March 18

Homework 8 to be assigned by Thursday, due Thursday, March 28

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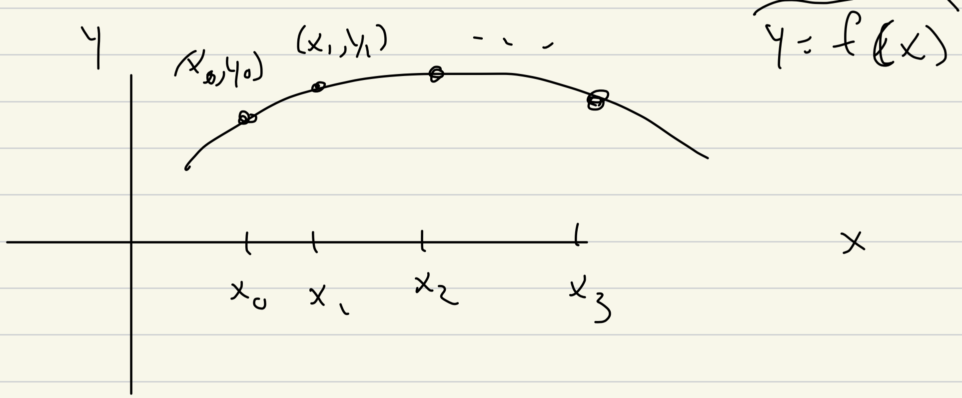
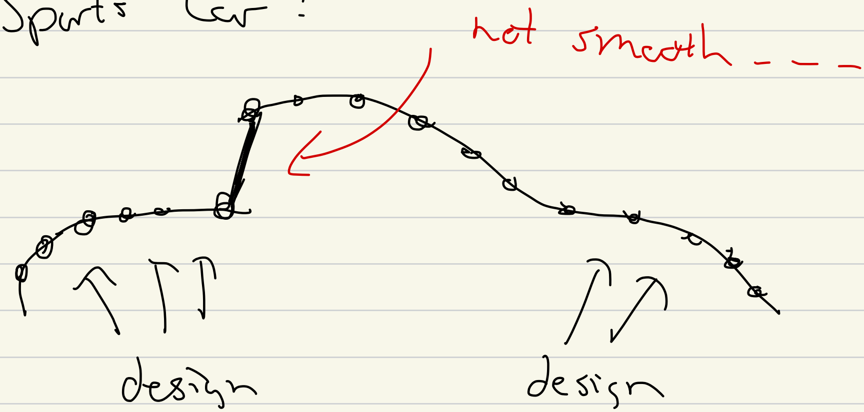
Chapter 11: <u>Splines</u>

(<u>Cubic</u> and related)




Want $\underline{v(x)}$ "near" $f(x)$

Sports Car!




these two parts have almost nothing to do with each other

$(x_0, y_0), \ldots \quad (x_n, y_n)$

$n$ quite large

$(x_0, y_0), (x_1, y_1)$ has little to do with $(x_n, y_n)$

– Interpolation is bad idea globally

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Approach:

Have $f = f(x)$, $x_0, \dots, x_n$,

$y_i = f(x_i)$, $f$ pretty smooth,

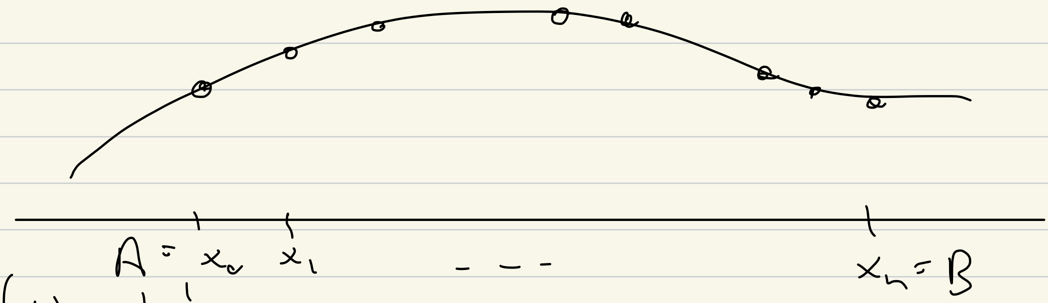
designed "locally", and you want

$v = v(x)$ that "looks like" what

$f$ should look like...

$f$ is measured exactly...

Think of




Want:

$v = v(x)$ such that

(1) $v(x_i) = y_i$ for all $i = 0, \dots, n$

(2) $v$ as smooth as is reasonable

Answer (Cubic spline) choose

$v$ so that

$$\mathcal{U} = \left\{ u \in C^{\text{some \# of derivatives}}(A, B) \quad \text{s.t.} \right.$$

$u(x_i) = y_i \;,\; i = 0, \dots, n \}$

and

$$\boxed{\mathcal{E}nergy_2(u) = \int_{x=A=x_0}^{x=B=x_n} \left(u''(x)\right)^2 dx}$$

is minimized at $v \in \mathcal{U}$ over all functions in $\mathcal{U}$

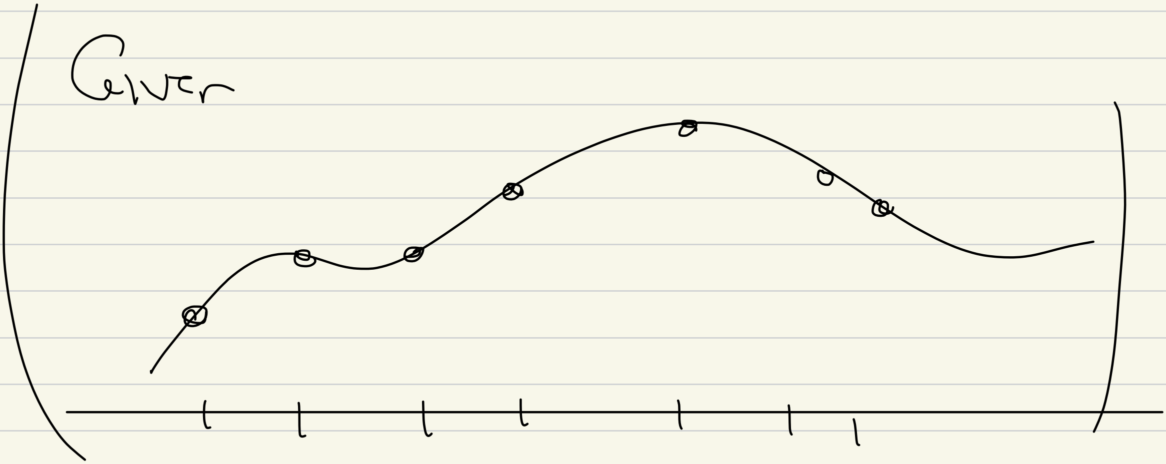
$\rightsquigarrow$ cubic splines

$$\boxed{\mathcal{E}nergy_1(u) = \int_{x=x_0}^{x=x_n} \left(u'(x)\right)^2 dx}$$

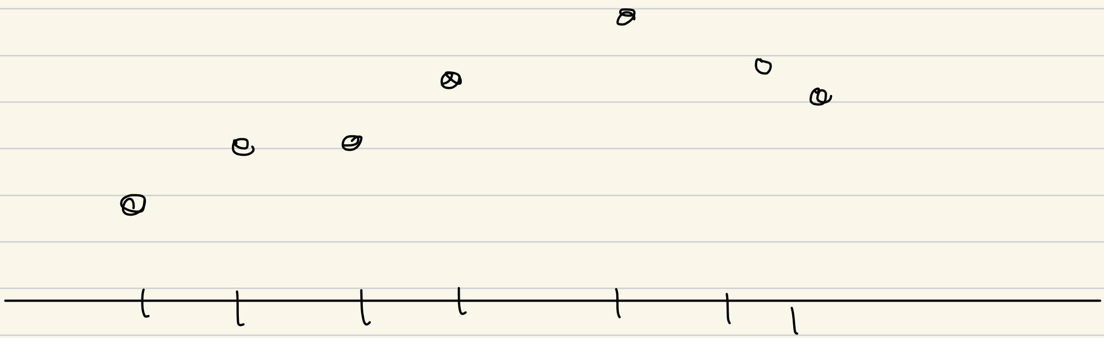
$$\boxed{\text{Length}(u) = \int_{x=x_0}^{x=x_n} \sqrt{1 + \left(u'(x)\right)^2}\, dx}$$

Other measures.

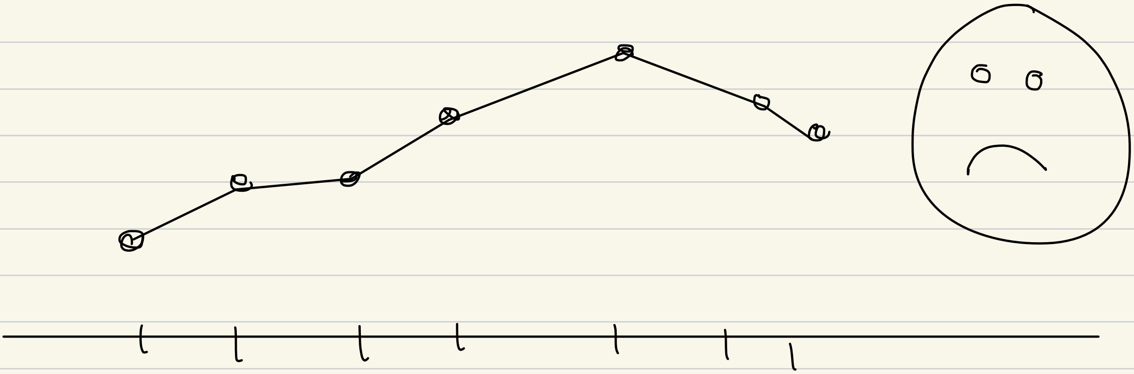
Given



We see:



We see:



Minimize Length$(u)$

$$\int_{x=x_0}^{x=x_n} \sqrt{1+\left(u'(x)\right)^2}\, dx$$

over $u \in \mathcal{U}$

$\hookrightarrow$ Piecewise Linear

"Dirichlet integral"

$$\mathcal{E}nergy_1(u) = \int_{x=x_0}^{x=x_n} (u'(x))^2 \, dx$$

Minimizer -- claim... is you'll

get $u$ = piecewise linear ☹

<u>Calculus</u> <u>of</u> <u>Variations</u>:

Say: $(v \in \mathcal{U})$ s.t.

$v$ goes thru $(x_i, y_i)$, $i = 0, \ldots, n$

and $\mathcal{E}nergy_1(u)$ is minimal...


take any $g \in C^\infty[x_0, x_1]$

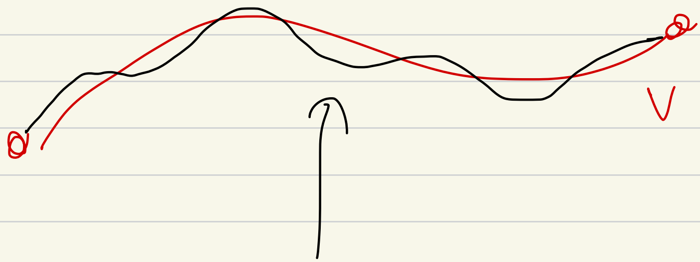
s.t. $\boxed{g(x_0) = 0, \quad g(x_1) = 0}$

Then

$v + g$ also goes thru $(x_0, y_0)$
$(x_1, y_1)$

Now look at

$$v_\varepsilon(x) = v(x) + \varepsilon\, g(x)$$




$$V_\varepsilon(x) = v(x) + \varepsilon\, g(x)$$

Then: $v \in \mathcal{U}$ minimizes

Energy: $\mathcal{U} \to \mathbb{R}$ :

$$\text{Energy}(v) \le \text{Energy}(V_\varepsilon)$$

$$= \text{Energy}(v + \varepsilon g)$$

$\varepsilon$ small f---

maybe Energy$(v) + \varepsilon$ $\square$

$$\text{Energy}_1(v+\varepsilon g)$$ ← minimized at $\varepsilon = 0$

$$= \int_{x=x_0}^{x=x_1} \left( v'(x) + \varepsilon g'(x) \right)^2 dx$$

$$= \int \left[ (v')^2 + 2\varepsilon(v')(g') + \varepsilon^2 (g')^2 \right]$$

$$= \text{Energy}(v)$$

$$+ \varepsilon \underbrace{\int 2(v')(g')\,dx}_{\text{implies must be } 0} + \varepsilon^2 \cdot \text{bounded}$$

$$\text{Energy}_1(v + \varepsilon g)$$

$$= \text{Energy}_1(v) + \varepsilon \int_{x=x_0}^{x=x_n} 2(v'(x))(g'(x))\,dx + \varepsilon^2 \int_{x=x_0}^{x=x_n} (g'(x))^2\,dx$$

$$= a + \varepsilon b + \varepsilon^2 c$$

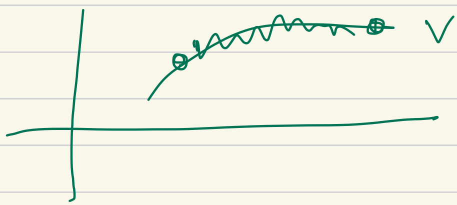
$$\Rightarrow b = 0 \quad \Rightarrow \int_{x=x_0}^{x=x_n} 2\, v'(x)\, g'(x)\,dx$$

must $= 0$

So ... for the "best" "lowest energy" $v$,

$$\int_{x=x_0}^{x=x_1} v'(x)\, g'(x)\, dx = 0$$

for any $g$ s.t. $g(x_0) = g(x_1) = 0$




int by part

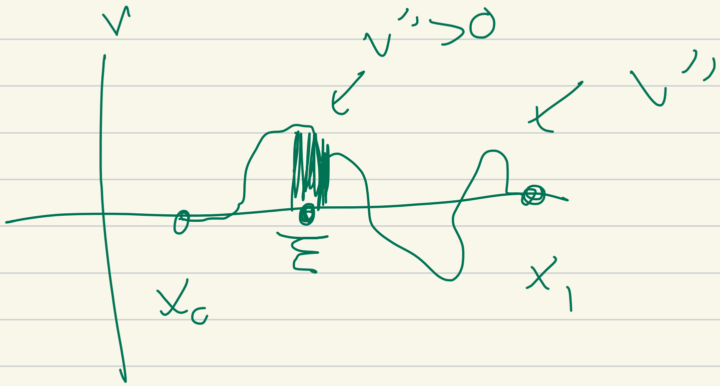
$$\int_{x=x_0}^{x=x_n} v'(x)\, g'(x)\, dx$$

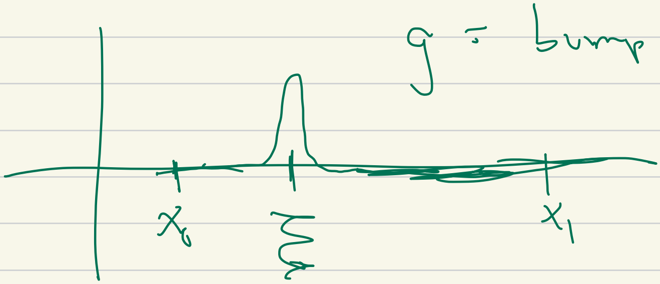
$$= \underbrace{v'(x)\, g(x) \Big|_{x=x_0}^{x=x_n}}_{=0} - \int_{x=x_0}^{x=x_n} v''(x)\, g(x)\, dx$$

$$\Rightarrow \int_{x=x_0}^{x=x_1} v''(x)\, g(x)\, dx \quad \text{must be } 0$$

hence $v''(x) = 0$

for all $x_0 < x < x_1$ :




$v'' > 0$ near $\xi$, $x_0 < \xi < x_1$

$\Longrightarrow$ $v''(\xi) = 0$ for all $x_0 < \xi < x_1$

$\Rightarrow$ $v$ is linear

from $x_0 \to x_1$

If $\text{Energy}_2(u)$ minimized at $u = v$

$$\int v'' g'' = 0$$

$$\to \int v'''' g = 0$$

$v'''' = 0 \Rightarrow v$ cubic