CPSC 303, March 13, 2024

- Today:
  §10.7 Hermite Interpolation

  ⇝ Ch 11: Splines (Piecewise Hermite Interpolation)

Say x_0 fixed as 2

$x_1 = 2+\varepsilon \qquad \varepsilon \to 0$

Fit $(x_0, f(x_0)), (x_1, f(x_1))$

with linear poly p:

as $\varepsilon \to 0$

Write

$$p(2) = f(2)$$

$$p(2+\varepsilon) = f(2+\varepsilon)$$

Monomial Interp:

$$p(x) = c_0 + c_1 x \;:$$

$$c_0 + 2c_1 = f(2)$$

$$c_0 + (2+\varepsilon)c_1 = f(2+\varepsilon)$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2+\varepsilon \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} f(2) \\ f(2+\varepsilon) \end{bmatrix}$$

Cond # $\begin{bmatrix} 1 & 2 \\ 1 & 2+\varepsilon \end{bmatrix} \longrightarrow \infty$

But ... linear ... calculation is easy ...

$$\boxed{\begin{aligned} 1 + 2c_0 &= f(2) \\ 1 + (2+\varepsilon)\, c_0 &= f(2+\varepsilon) \end{aligned}}$$

$$\boxed{1 + 2c_0 = f(2)}$$

$$\varepsilon c_0 = f(2+\varepsilon) - f(2)$$

$$\boxed{c_0 = \frac{f(2+\varepsilon) - f(2)}{\varepsilon}}$$

fixed

as $\varepsilon \to 0$

$$\boxed{\begin{aligned} 1 + 2c_0 \\ c_0 \end{aligned}} \begin{aligned} &= f(2) \longrightarrow f(2) \\ &= \frac{f(2+\varepsilon) - f(2)}{3} \to f'(2) \end{aligned}$$

What about $\qquad\qquad\qquad\qquad \varepsilon \to 0$

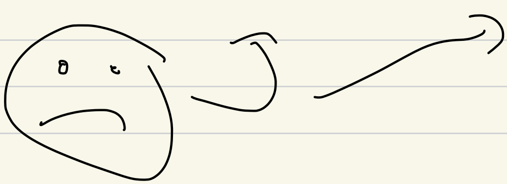
$$\left.\begin{array}{l} x_0 = 2, \; x_1 = 2+\varepsilon \\ x_2 = 4, \; x_3 = 4+\varepsilon \end{array}\right\} \text{Hermite interpolate at 2 and 4}$$

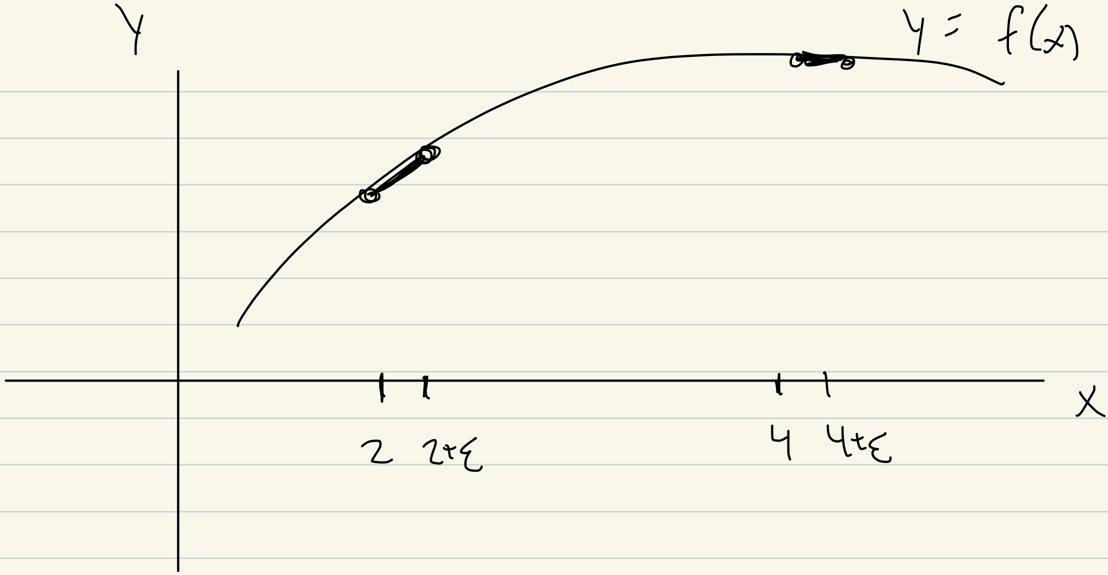
monomial!

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

$\varepsilon \to 0$

$$\begin{bmatrix} 1 & 2 & 2^2 & 2^3 \\ 1 & 2+\varepsilon & (2+\varepsilon)^2 & (2+\varepsilon)^3 \\ 1 & 4 & 4^2 & 4^3 \\ 1 & 4+\varepsilon & (4+\varepsilon)^2 & (4+\varepsilon)^3 \end{bmatrix} \vec{c} = \begin{bmatrix} f(2) \\ f(2+\varepsilon) \\ f(4) \\ f(4+\varepsilon) \end{bmatrix}$$

$\varepsilon \to 0 \to$ 



$\varepsilon \to 0$ we'd guess we get!

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

but ...

$$\boxed{\begin{array}{ll} p(2) = f(2), & p(4) = f(4) \\ p'(2) = f'(2), & p'(4) = f(4) \end{array}}$$

Hermite Interpolation: Interpolation, but we allow derivative agreement

$$\left.\begin{aligned}
c_0 + c_1 2 + c_2 2^2 + c_3 2^3 &= f(2) \\
c_1 + 2c_2 \cdot 2 + 3c_3 \cdot 2^2 &= f'(2) \\
c_0 + c_1 4 + c_2 4^2 + c_3 4^3 &= f(4) \\
c_1 + 2c_2 \cdot 4 + 3c_3 \cdot 4^2 &= f'(4)
\end{aligned}\right\}$$

$$\left[\begin{aligned}
&(c_0 + c_1 x + c_2 x^2 + c_3 x^3)' \\
&\quad = c_1 + 2c_2 x + 3c_3 x^2
\end{aligned}\right]$$

- Lin Alg w/o Lin Alg
- Divided Differences
- Less usable: Monomial & Lagrange

Look at homogeneous form of 4x4:

$$\boxed{c_0 + c_1 2 + c_2 2^2 + c_3 2^3} = 0 \qquad p(2)$$

$$\boxed{c_1 + 2c_2 \cdot 2 + 3c_3 \cdot 2^2} = 0 \qquad p'(2)$$

$$c_0 + c_1 4 + c_2 4^2 + c_3 4^3 = 0$$

$$c_1 + 2c_2 \cdot 4 + 3c_3 \cdot 4^2 = 0$$

What would this mean for

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \quad ?$$

This would mean

$$p(2),\ p'(2),\ p(4),\ p'(4) = 0$$

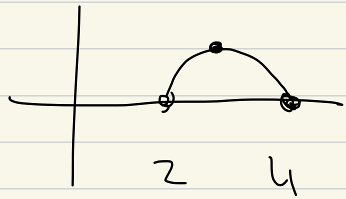
Thm: If $p(x)$, deg ≤ 3, polynomial,

and $p(2) = p'(2) = p(4) = p'(4) = 0$

then p is the zero polynomial.

Pf:

$$p(2) = p(4) = 0$$



$\Rightarrow$ $2 < \xi < 4$ s.t.

$$p'(\xi) = 0$$

but

$p'(2) = 0$ $\qquad\qquad$ $p'(4) = 0$

p'' has zeros between $2, \xi$ and $\xi, 4$

p''' " " " somewhere

$$p'''(x) = \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3\right)'''$$

$$= \quad 0 \qquad 0 \qquad 0 \qquad 3 \cdot 2 \cdot 1 \cdot c_3$$

so $c_3 = 0$

so $p(x) = c_0 + c_1 x + c_2 x^2$

$p''(x)$ has a zero

$$\left(c_0 + c_1 x + c_2 x^2\right)''$$

$$0 \qquad 0 \qquad 2 \cdot 1 \cdot c_2$$

has a zero, so $c_2 = 0$

Similarly $c_1 = 0$, $c_0 = 0$

Divided diff:

$$f[2, 2+\varepsilon] = f'(\xi)$$

$$2 \leq \xi \leq 2+\varepsilon$$

$$f(2,2] = \lim_{\varepsilon \to 0} f(2, 2+\varepsilon)$$

$$= \lim_{\varepsilon \to 0} f'(\xi)$$

$$2 < \xi < 2+\varepsilon$$

$$= f'(2)$$

$$f[4,4] = \lim_{\substack{x_0 \to 4 \\ x_1 \to 4}} f[x_0, x_1) = f'(4)$$

So:

$$p(x) = f[2] + f[2, 2+\varepsilon](x-2)$$

$$+ f[2, 2+\varepsilon, 4](x-2)(x-(2+\varepsilon))$$

$$+ f[2, 2+\varepsilon, 4, 4+\varepsilon](x-2)(x-2-\varepsilon)(x-4)$$

as $\varepsilon \to 0$, if f' exists and is continuous, has a limit

$$\overset{\varepsilon \to 0}{=} f[2] + f[2,2](x-2)$$

$$+ f[2,2,4](x-2)^2$$

$$+ f[2,2,4,4](x-2)^2(x-4)$$

$$f[2, 2+\varepsilon, 4]$$

$$= \frac{f[2+\varepsilon, 4] - f[2, 2+\varepsilon]}{4-2}$$

$\varepsilon \to 0$

$$\longrightarrow \frac{f[2,4] - f[2,2]}{2}$$

Homework 7:

$$\begin{bmatrix} 1 & 2 \\ 1 & 2+\varepsilon \end{bmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \end{bmatrix}$$

$$\operatorname{cond}\left(\begin{bmatrix} \ \end{bmatrix}\right) \simeq \frac{1}{\varepsilon} \cdot C$$

Lagrange:

$$\boxed{y_0 \cdot \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}}$$

$y_0 = 1,\ y_1 = 0 \rightarrow$ Lagrange

$y_0 = 0,\ y_1 = 1 \rightarrow$ "

$y_0 = f(x_0),\ y_1 = f(x_1) = f(x_0 + \varepsilon)$

$$y = \alpha_0 \frac{x - x_1}{x_0 - x_1} + \alpha_1 \frac{x - x_0}{x_1 - x_0}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} ''$$

$$'' \begin{pmatrix} 1 & x_0 \\ 1 & x_0 + \varepsilon \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} ''$$

(7) HW:

$$x_0 = 1, \quad x_1 = 1/2,$$

x_2 exact with $\left(\frac{1}{4}\right)$

$\vdots$

$$x_{n+2} = \left(\frac{3}{2}\right)\underbrace{\left(x_{n+1}\right)} - x_n$$

$$2^{-1073}$$

 $\underbrace{\left(\frac{3}{2}\right)\left(2^{-1074}\right)}$

(7b)

$x_0 = 1, \quad x_1 = 1/4$

$$x_n = \underbrace{c_1}_{-1/2} \cdot 1^n + \underbrace{c_2}_{\text{whatever}} \left(\tfrac{1}{2}\right)^n$$

approx

$$\boxed{-.500\ldots + (\to 0)}$$

roughly 10^{-16} relative error

roughly 10^{-18} rel err

$$\frac{3}{2} x_{n+1} - \frac{1}{2} x_n$$

$(.50\ldots + \text{small})$ $(.500\ldots + \text{small})$

$$x_{n+2} = \frac{3}{2} x_{n+1} - \frac{1}{2} x_n$$

$$x_{n+2} - \frac{3}{2} x_{n+1} + \frac{1}{2} x_n = 0$$

σ, r

$$r^2 - \frac{3}{2} r + \frac{1}{2} = 0$$

$$(r - 1/2)(r - 1) = 0$$

$r = 1/2,$

$r = 1$