CPSC 303, March 11, 2024

Topics related to divided differences

- $f[x_0,\ldots,x_n] = \dfrac{f[x_1,\ldots,x_n]-f[x_0,\ldots,x_{n-1}]}{x_n-x_0}$

proven in

"CPSC 303: Remarks on Divided Differences (2024)"

- Change of basis:

$1, x, x^2 \longleftrightarrow 1, x-1, (x-1)(x-2)$

lower/upper triangular systems

- Error in interpolation, Chebyshev Interpolation (10.6)

- Degenerate Interpolation

  - Taylor's Theorem

  - Hermite Interpolation $\rightsquigarrow$ Splines (Ch. 11)

- Midterm will here ESB 1012 on Friday

Last time:

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

is non-trivial

you can do this using Lagrange interpolation formulas.

Textbook

$$\begin{tikzcd}
 & & f[x_0] \arrow[dl] \\
 & f[x_0,x_1] \arrow[dl] & \\
f[x_0,x_1,x_2] & & f(x_1) \arrow[ul] \arrow[dl] \\
 & f[x_1,x_2] \arrow[ul] & \\
 & & f[x_2] \arrow[ul]
\end{tikzcd}$$

Textbook: To compute

$$f[x_i], \quad f[x_i, x_{i+1}], \quad f[x_i, x_{i+1}, x_{i+2}],$$

$$\ldots$$

$x_0, \ldots, x_n$:

$$\begin{array}{ll} n+1 & f[x_i] \\ n & f[x_i, x_{i+1}] \\ n-1 & f[x_i, x_{i+1}, x_{i+2}] \\ \vdots & \\ 1 & f[x_0, \ldots, x_n] \end{array}$$

total time $\approx n^2$

Comparison:

Lagrange Inter $\approx n^2$

omit i

$$\sum f(x_i) \left(\frac{(x-x_0) \cdots (x-x_n)}{(x_i-x_0) \cdots (x_i-x_n)} \right)$$

omitting $\dfrac{x-x_i}{x_i-x_i}$

$O(n^2)$ $\underbrace{\text{floating point operations}}_{\text{"flops"}}$

Monomial:

$$\begin{bmatrix} & & \\ & & \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

roughly $O(n^3)$ operations

One point about

upper / lower triangular

matrices & change of bases!

poly deg ≤ 2

$$= \{ \alpha_0 + \alpha_1 x + \alpha_2 x^2 \}$$

Newton dN diff: $x_0 = 1, \ x_1 = 3$

same

poly deg ≤ 2: $\{ c_0 + c(x - x_0) + c(x-x_0)(x-x_1) \}$

e.g. $\{ c_0 + c_1(x-1) + c_2(x-1)(x-3)$

Rem:

$$1 = 1$$

$$x-1 = x\cdot 1 + (1)(-1)$$

$$(x-1)(x-3) = x^2 - 4x + 3$$

$$= x^2(1) + x(-4) + 3$$

$$\begin{bmatrix} 1 \\ x-1 \\ (x-1)(x-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ x-1 \\ (x-1)(x-3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$ is lower triangular matrix,

hence invertible...

More generally $q_0(x), q_1(x), q_2(x)$

s.t. $$q_0(x) = \beta_0 \neq 0$$

$$q_1(x) = \beta_1 x + \text{lower}$$

$$q_2(x) = \beta_2 x^2 + \text{lower}$$

$\beta_0, \beta_1, \beta_2 \neq 0$, then you can convert from $1, x, x^2$ to q_0, q_1, q_2

$1, x, x^2$ can write as

$1 \cdot$ something $+ (x-1)$ something etc

$+ (x-1)(x-3)$ something diff

$1 = 1$

$x = (x-1) + 1$

$x^2 = \underbrace{(x-1)(x-3)}_{x^2-4x+3} + 4x - 3$

$3 = 3$

$4x = 4(x-1) + 4$

$$1 = 1$$

$$x = (x-1) + 1$$

inductively

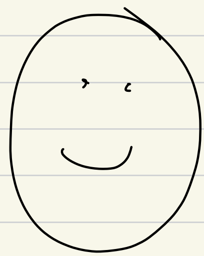
$$x^2 = (x-1)(x-3) + 4(x-1) + 4 - 3$$

$$= (x-1)(x-3) + 4(x-1) + 1$$

$$\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x-1 \\ (x-1)(x-3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Lagrange basis: $x_0 = 1$, $x_1 = 3$, $x_2 = 5$

polys: $\dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \longrightarrow$ quadratic

and both others ---

$$\begin{pmatrix} 1 \\ x-1 \\ (x-1)(x-3) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

Note

$$\begin{pmatrix} \text{function}_1(x) \\ \text{function}_2(x) \\ \text{funct}_3(x) \end{pmatrix} = \begin{bmatrix} \text{matrix} \\ \text{of} \\ \text{scalars} \\ (\mathbb{R}) \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

vector of functions

Interpolation Error: Idea:

$f(x)$ agrees with poly $p_n(x)$ on

data points $x_0, \dots, x_n$:

$$\boxed{p_n(x) = c_0 + c_1(x-x_0) + \dots + c_n(x-x_0)\dots(x-x_n)}$$

and on $x_0, \dots, x_n, x_{n+1}$ for

$$\boxed{p_{n+1}(x) = p_n(x) + \underbrace{c_{n+1}}_{f[x_0, \dots, x_{n+1}]} \underbrace{(x-x_0)\dots(x-x_n)}_{\text{tricky expr.}}}$$

Now consider x_{n+1} to be a variable

$x \leftarrow$ variable ...

How to get Taylor's thm from divided differences ...

Newton's formula!

If

$$f(x_{n+1}) = p_{n+1}(x_{n+1})$$

$$f(x) = p(x)$$

$$f(x) = p_{n+1}(x)$$

$$= c_0 + c_1(x-x_0) + \dots + c_n(x-x_0)\dots(x-x_{n-1})$$

$$+ f[x_0, \dots, x_n, x](x-x_0)\dots(x_0-x_n)$$

We know

$$f[x_0, \ldots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

where $\xi \in$ interval containing $x_0, \ldots, x_n, x$

__Thm__:

$$f(x) = \overset{f[x_0]}{c_0} + \overset{f[x_0,x_1]}{c_1}(x-x_0) + \ldots + \overset{f[x_0,\ldots,x_n]}{c_n}(x-x_0)\ldots(x-x_{n-1})$$

$$+ \frac{f^{(n+1)}(\xi)}{n!}(x-x_0)\ldots(x-x_n)$$

Taylors Thm: $x_1, x_2, \ldots, x_n \longrightarrow x_0$

Then:

$$f(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2$$
$$+ c_3(x-x_0)^3 + \dots + c_n(x-x_0)^n$$
$$+ \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},$$

each

$$c_i = \frac{f^{(i)}\left(\underset{x_0, x_1, \dots, x_i}{\text{point in interval}}\right)}{(i+1)!}$$

$$\xrightarrow{x_1, \dots, x_n \to x_0} \frac{f^{(i)}(x_0)}{(i+1)!}$$

Double precision . . .

smallest pos # in double precision

is the subnormal number

$$2^{-1074}$$

$$\underbrace{0.00\ \_\ \_\ \_\ 01}_{52 \text{ bits}} \qquad 2^{-1022}$$

next smallest: $2 \cdot 2^{-1074}$

$3 \cdot 2^{-1074}$

$\vdots$