CPSC 303 March 8, 2024

- Remarks on exam
- Divided differences:

(1) Finish proof that

$$p(x) = f[x_0] + (x-x_0) f[x_0,x_1] + \cdots + (x-x_0)\cdots(x-x_{n-1}) f[x_0,\ldots,x_n]$$

interpolates

$$(x_0, y_0 = f(x_0)), \ldots, (x_n, y_n = f(x_n))$$

(2) Upper triangular systems

(3) Degeneracy

- Exam Fri, March 15 is HERE.
- Sample midterm problems are on course website
- Take questions!
  last 1/2 of class Mon, Wed

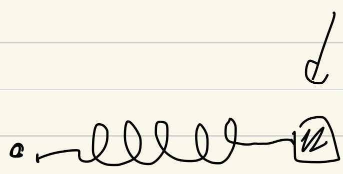
2-sides of one 8.5" × 11" sheet of paper --

Trick / alt. methods:

$$m\ddot{x} = -Cmx$$

$x(t)$

$x: \mathbb{R} \to \mathbb{R}$



Mult by $\dot{x}$

$$m\ddot{x}\dot{x} = -Cmx\dot{x}$$

OR

$$m\ddot{x} = x^5$$

$$m\ddot{x}\dot{x} = (x^5)\dot{x}$$

Trick: $\ddot{x}\dot{x}$ simply: $\dot{\left(\frac{1}{2}(\dot{x})^2\right)}$

$$\frac{d}{dt}(\dot{x})^2 = 2\dot{x}\left(\frac{d}{dt}\dot{x}\right) = 2\dot{x}\ddot{x}$$

$m\ddot{x}\dot{x}$ it's also $\underbrace{\frac{d}{dt}\left(\frac{1}{2}m(\dot{x})^2\right)}$

$\frac{1}{2}mv^2$

= Kinetic

if

$$\underbrace{m\ddot{x}\dot{x}} = (x^5)\dot{x}$$

$$\frac{d}{dt}\left(\frac{1}{2}(\dot{x})^2\right) = \underbrace{\frac{d}{dt}\left(\frac{1}{6}x^6\right)}_{\frac{1}{6}(6x^5)\dot{x}}$$

So

$$m\ddot{x} = x^5$$

$\Rightarrow$

$$m\ddot{x}\,\dot{x} = x^5\,\dot{x}$$

$$\frac{d}{dt}\left(\frac{1}{2}m(\dot{x})^2\right) = \frac{d}{dt}\left(\frac{1}{6}x^6\right)$$

$$\frac{1}{2}m(\dot{x})^2 = \frac{1}{6}x^6 + C$$

$$\underbrace{\frac{1}{2}m(\dot{x})^2}_{\text{K.E.}} - \underbrace{\frac{1}{6}x^6}_{\text{P.E.}} = \underbrace{C}_{\text{const}}$$

Exam!

$$\|\vec{x}\|_\infty$$

$$\|\vec{x}\|_2$$

$$\vdots$$

Matrix!

$$\|A\|_\infty, \quad \|A^{-1}\|_\infty$$

$$\text{cond}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

$$\left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\|_\infty = \max\left(|a| + |b|, \; |c| + |d| \right)$$

So we know

$$\left\| \begin{bmatrix} 3 & 7 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_\infty$$

$$\leq \underbrace{\left\| \begin{bmatrix} 3 & 7 \\ -1 & 8 \end{bmatrix} \right\|_\infty}_{\underbrace{\max(3+7,\ 1+8)}_{10}} \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_\infty$$

Also

$$\left\| \begin{bmatrix} 3 & 7 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|$$ can equal

$$10 \cdot \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_\infty$$

will equal for either

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

or

$$\text{''} \quad \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}$$

$$\left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_\infty = \max(|x_1|, |x_2|)$$

also

$$\lim_{p \to \infty} \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_p$$

$$= \lim_{p \to \infty} \left(|x_1|^p + |x_2|^p \right)^{1/p}$$

To be continued ---

Divided differences:

$p_{n-1}(x)$ fits $(x_0, f(x_0)), \dots, (x_{n-1}, f(x_{n-1}))$

add $(x_n, f(x_n))$

see we need $p_n(x)$ deg $\le n$

$$p_n(x) = p_{n-1}(x) + c_n \underbrace{(x-x_0)\cdots(x-x_{n-1})}_{\begin{cases} 0 & \text{for } x = x_0, \dots, x_{n-1} \\ \neq 0 & \text{for } x = x_n \end{cases}}$$

Hence

$$c_n = \left(\frac{p_n(x) - p_{n-1}(x)}{(x-x_0)\cdots(x-x_{n-1})} \right) \Bigg|_{x=x_n}$$

$$= \frac{\underbrace{p_n(x_n)}_{f(x_n)} - \underbrace{p_{n-1}(x_n)}_{\text{compute}}}{(x_n-x_0)\cdots(x_n-x_{n-1})}$$

Start:

To fit $(x_0, y_0) = (x_0, f(x_0))$

$p_0(x) = \text{const} = f(x_0)$

p_0 is of deg ≤ 0, $p_0 = \text{const} = f(x_0)$

fit $(x_1, f(x_1))$:

$$p_1(x) = p_0(x) + c_1 (x - x_0)$$

for $x = x_1$

$$\underbrace{p_1(x_1)}_{\substack{f(x_1) \\ \text{or} \\ y_1}} = f(x_0) + c_1 (x_1 - x_0)$$

$$c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= f[x_0, x_1]$$

$$p_2(x) = p_1(x) + c_2 (x - x_0)(x - x_1)$$

fit $(x_2, f(x_2))$ or (x_2, y_2)

$$c_2 = \frac{p_2(x_2) - p_1(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2) - p_1(x_2)}{(x_2 - x_0)(x_1 - x_0)}$$

$$= \frac{f(x_2) - \left[f(x_0) + f[x_0, x_1](x_2 - x_1) \right]}{(x_2 - x_0)(x_1 - x_0)}$$

$= \ldots ?$

$$= \frac{f(x_2) - \left(f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x_2 - x_0) \right)}{(x_2 - x_0)(x_2 - x_1)}$$

$= \ldots ??$

f) isolate $f(x_0) = y_0$, $f(x_1) = y_1$, $f(x_2) = y_2$

$$= \frac{f(\cancel{x}_2)}{(x_2 - x_0)(x_2 - x_1)} + \underbrace{\frac{-f(x_1)\,\cancel{(x_2 - x_0)}}{(x_1 - x_0)(x_2 - x_1)\cancel{(x_2 - x_0)}}}_{\frac{f(x_1)}{(x_1 - x_0)(x_2 - x_1)}}$$

+ $f(x_2)$ term

$$= \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} + \frac{-f(x_1)}{(x_1-x_0)(x_2-x_1)}$$

$+ f(x_0)$ term

$$\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}$$

$+ f(x_2)$ term

Symmetry

looks like

$$\sum_{i=0,1,2} \frac{f(x_i)}{\prod_{j\neq i}(x_i-x_j)} \quad (*)$$

This can be inferred via

Lagrange interpolation ...

remarkable obs

$$\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

$$= f[x_0,x_1,x_2]$$

long calc.

Using (*) we can also prove

$$f[x_0,x_1,x_2,x_3] = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$$

Monday: Proof of Taylor's

thm ...