CPSC 303, March 6, 2024

Midterm = March 15, Location: TBA

No new homework, instead

Review/Sample Problems: this week

Next week: Part of Monday
" " Wednesday } Questions

Today: 2 stories about Ch 10,

first is partially review ...

Divided differences: long story --

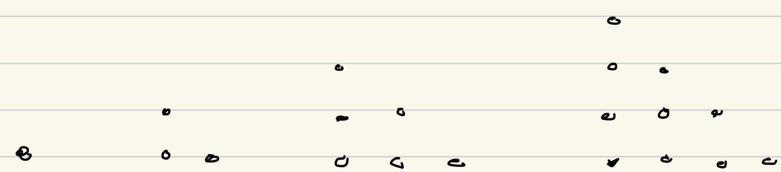
we'll get to some of the story.

Connection with polynomial interpolation

is more recent --

Classical story:

Triangular Numbers:



1   3   6   10   dots

$x_0 = 1$, $x_1 = 2$, $x_3 = 3$, $x_4 = 10$, ...

triagle size

$x_0 = 1$ $x_1 = 2$ $x_2 = 3$ $x_3 = 4$, Size of Triangle

1 3 6 10, ..., # dots

1, 3, 6, 10, 15, 21, 28, ...

expirement could have some error ...

1 3 6 10 15 21 28

y_0

$x_0 = 1$ ....

	1	3	6	10	15	21	28
diff	2	3	4	5	6	7	
diff^2	1	1	1	1	1		

diff^2 $\quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$

diff^3 $\quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad ---$

$y_0 = 1, \; y_1 = 3, \; y_2 = 6, \; y_3 = 10, \; y_4 = 15, \; ---$

$2 \qquad 3 \qquad 4$

$y_1 - y_0 \qquad y_2 - y_1 \qquad y_3 - y_2$

$$(\sigma - 1)(y_n) = \sigma(y_n) - (y_n)$$

$$= y_{n+1} - y_n$$

$$\left.\begin{array}{l}\text{"diff"}\\ \text{"difference"}\end{array}\right\}\quad \sigma - 1$$

(diff)^2 $\qquad (\sigma - 1)^2$

(diff)^3 $\qquad (\sigma - 1)^3$

We knew

$(\sigma - 2)^3 (y_n) = 0$ ans $y_n = 2^n p(n)$

$\deg p \leq 2$

$(\sigma - 1)^3 (y_n) = 0$

ans $y_n = 1^n p(n)$

$= p(n)$

p ~~p~~ poly deg ≤ 2

So $(r-1)^3 (y_n) = 0$

we knew

$$y_n = a + bn + cn^2$$

from ODE/Recurrences

$y_0 = 1$ $y_1 = 3$ $y_2 =$ [illegible] $y_3 = 10$ $y_4 = 15, \ldots$

???

Data 1, 3, $\cancel{6}$, 10, 15, 21, $\cancel{28}$, 36

2 7?? 5 6 15??

Some not equally spaced

$t_0 = 1 \quad t_1 = 2 \quad t_2 = 4 \quad t_3 = 5 \quad 6 \quad 8$

$y_0 = 1 \quad y_1 = 3 \quad y_2 = 10 \quad 15 \quad 21 \quad 36$

$f(t_2) - f(t_1)$

$\frac{3-1}{2}$ $\quad$ $\frac{10-3=7}{2}$ $\leftarrow y_2 - y_1$

$2 \leftarrow t_2 - t_1$

time step

What to do? Ans: Interpolation

Yes!

$$\frac{f(t_{i+1})-f(t_i)}{t_{i+1}-t_i} \quad \left(\text{recall } f[t_i, t_{i+1}]\right)$$

missing

1 3 10 15 21 36

$$\frac{3-1}{1} \quad \frac{10-3}{2} \quad \frac{15-10}{1} \quad \frac{21-15}{1} \quad \frac{36-21}{2}$$

$$2 \quad 7/2 \quad 5 \quad 6 \quad 15/2$$

$$f[t_i, t_{i+1}]$$

Remind! In interpolation:

$$c_2 = f[x_0, x_1, x_2]$$

$$= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$f[t_0,t_1]$	$f[t_1,t_2]$	$f[t_2,t_3]$	$f[t_3,t_4]$
2	7/2	5	6

$$\frac{f[t_1,t_2] - f[t_0,t_1]}{t_2 - t_0} \leftarrow \text{time step}$$

$t_0=1$ $\quad t_1=2$ $\quad t_2=4$ $\quad t_3=5$ $\quad t_4=6$ $\quad t_5=8$

$2 \quad 7/2 \quad 5 \quad 6 \quad 11/2$

$$\frac{7/2-2}{4-1} \quad \frac{5-7/2}{5-2} \quad \frac{6-5}{6-4} \quad \frac{15/2-6}{3}$$

$$\frac{3/2}{3} \quad \frac{3/2}{3} \quad \frac{1}{2} \quad \frac{3/2}{3}$$

$$\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}$$

$$0 \quad 0 \quad 0$$

<u>Thm</u>: If given $(t_0, y_0), (t_1, y_1) \ldots$

and $y_i = p(t_i)$, p deg $\leq d$

Then $(d+1)^{\text{st}}$ divided differences

we get 0, when

$$\begin{array}{llll} t_0 & t_1 & t_2 & \\ y_0 & y_1 & y_2 & \cdots \end{array}$$

$$f[t_i] = y_i$$

$$f[t_i, t_{i+1}] = \frac{f[t_{i+1}] - f[t_i]}{t_{i+1} - t_i}$$

and

$$f[t_i, t_{i+1}, \ldots, t_j] =$$

$$= \frac{f[t_{i+1}, \dots, t_j] - f[t_i, \dots, t_{j-1}]}{t_j - t_i}$$

$=$

Def (Classical) Also [A & G] :

Given $f: \mathbb{R} \to \mathbb{R}$, $t_0, t_1, \dots, t_n$

we define the Newton divided

differences as above

$$f[t_i] = f(t_i) \qquad (= y_i)$$

$$f[t_i, t_{i+1}] = \frac{f[t_{i+1}] - f(t_i]}{t_{i+1} - t_i}$$

$$f[t_i, t_{i+1}, t_{i+2}] = \frac{f[t_{i+1}, t_{i+2}] - f[t_i, t_{i+1}]}{t_{i+2} - t_{i+1}}$$

etc.

Theorem [A&G]:

If $p_{n-1}(x)$ deg $n-1$ s.t.

$p(x_0) = f(x_0)$, $p(x_1) = f(x_1)$, ...

$p(x_{n-1}) = f(x_{n-1})$

for $x_0, x_1, \ldots$ and you

want

$p_n(x)$ to fit $(x_0, f(x_0))$, ...

$(x_{n-1}, f(x_{n-1}))$,

$(x_n, f(x_n))$

Then

$$p_n(x) = p_{n-1}(x) +$$

$$(x-x_0)(x-x_1)\cdots(x-x_{n-1})\cdot C_n$$

and

$$C_n \overset{\text{classical}}{=} f[x_0, x_1, \ldots, x_n].$$

So

$p_n(x)$

$$= f[x_0] + (x-x_0)\, f[x_0, x_1] +$$

$$(x-x_0)(x-x_1)\, f[x_0, x_1, x_2] + \cdots$$

$$(x-x_0)(x-x_1)\cdots(x-x_{n-1})\, f[x_0, \ldots, x_n]$$

Class ends

Student remark !

if we knew

$$f[x_0, \dots, x_n] = f^{(n)}(\xi)$$

and f is poly deg $\leq n-1$,

then

$$f[x_0, \dots, x_n] = 0 \quad \text{for all } x_0, \dots, x_n$$

This proves the theorem.