CPSC 303, March 4, 2024

Last time: Quadratic curve fitting:

$$p(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1)$$

OR $\quad f[x_0] \quad f[x_0, x_1] \quad f[x_0, x_1, x_2]$

s.t.
$$\begin{array}{r} f(x_0) = \\ f(x_1) = \\ f(x_2) = \end{array} \boxed{\begin{array}{lcl} y_0 & = & p(x_0) \\ y_1 & = & p(x_1) \\ y_2 & = & p(x_2) \end{array}}$$

Data $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

Imagine: fitting $p(x)$ to $f(x)$

Monomial: $p(x) = \hat{c}_0 + \hat{c}_1 x + \hat{c}_2 x^2$

Rather then $1, x, x^2$

Use: $1, \quad x - x_0, \quad (x-x_0)(x-x_1)$

$$\left.\begin{array}{l} \text{Adaptive: data } (x_0, y_0) \\ \\ \text{fit } p_0(x) \text{ deg} \leq 0 \quad \text{const} \\ \\ p_0(x) = \text{const} = y_0 \end{array}\right\} \text{Step 0}$$

Get (x_1, y_1) new poly

$p_1(x)$ fit old pts (x_0, y_0)

and new pt (x_1, y_1)

$$p_1(x) = p_0(x) + c_1 (x - x_0)$$

at x_0, $p_1(x_0) = p_0(x_0) + c_1 (x_0 - x_0) = p_0(x_0)$

$$\underbrace{p_1(x_1)}_{} = \underbrace{p_0(x_1)}_{\text{whatever}} + c_1 \underbrace{(x_1 - x_0)}_{\text{whatever, but } \neq 0}$$

$$c_1 = \frac{p_1(x_1) - p_0(x_1)}{x_1 - x_0}$$

$p_1(x_1) = y_1$ want!

$$c_1 = \frac{y_1 - p_0(x_1)}{x_1 - x_0} \quad \checkmark$$

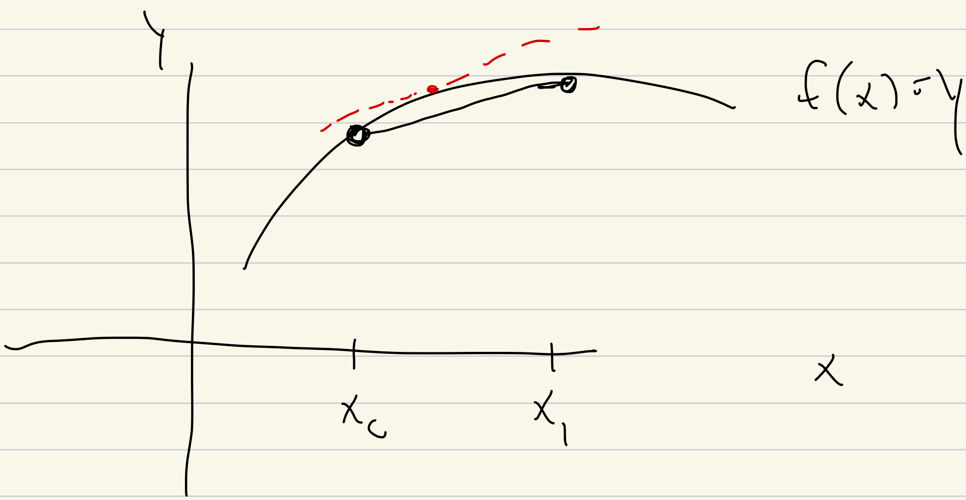
$$\boxed{\begin{array}{l}\text{Claim: } \quad y_0 = f(x_0),\ y_1 = f(x_1) \\ c_1 = f'(\xi), \text{ where } \xi \text{ is} \\ \text{on interval containing } x_0, x_1\end{array}}$$

$$c_1 = \frac{y_1 - p_0(x_1)}{x_1 - x_0}$$

$y_1 = f(x_1)$

$p_0(x_1)$
$= f(x_0)$

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Mean Value Theorem (Taylor's Thm)

$$c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi)$$

Next step:

$$p_1(x) = c_0 + c_1(x - x_0)$$

$f(x_0)$

depends only on x_0, f

$f[x_0]$

$\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

depends only on x_0, x_1, f

$f[x_0, x_1]$

Now

$$p_2(x) = \underbrace{p_1(x)}_{c_0 + c_1(x - x_0)} + c_2(x - x_0)(x - x_1)$$

We'll call c_2 : depends x_0, x_1, x_2

f

$(f(x_0), f(x_1), f(x_2))$

$f[x_0, x_1, x_2]$

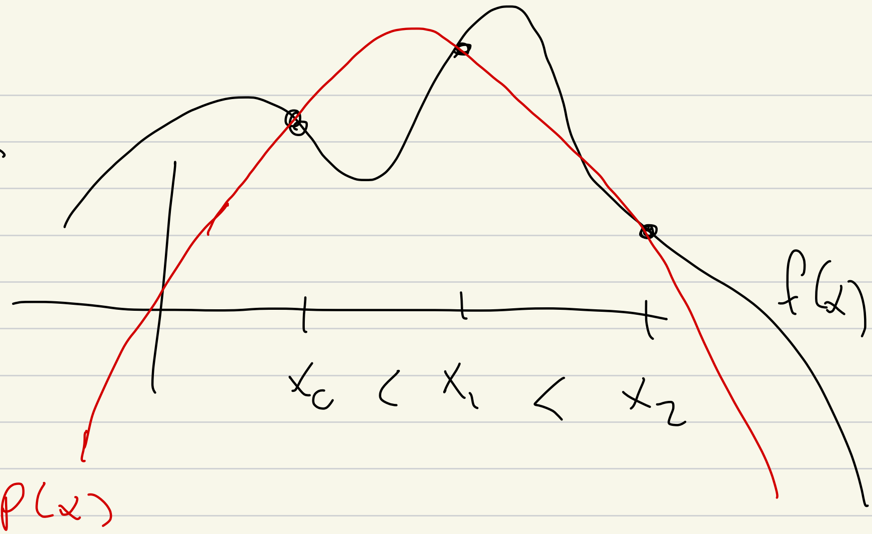
Thm: If I interval containing x_0, x_1, x_2

and f is twice differentiable on I,

then $\exists\, \xi \in I$ s.t.

$$f[x_0, x_1, x_2] = \frac{1}{2} f''(\xi)$$

Proof:



$g(x) = f(x) - p(x)$ has zeros at

x_0, x_1, x_2

$$g(x_0) = 0 \qquad g(x_1) = 0 \qquad g(x_2) = 0$$

Rolle's thm

$$g'(\xi_1) = 0 \qquad g'(\xi_2) = 0$$

$$g''(\eta) = 0$$

$$g(x) = f(x) + c_2 x^2 + \text{lower order}$$

$x, 1$

$$\tilde{C}_2 X^2 + \tilde{C}_1 X + \widehat{C}_0 = c_2 x^2 + \text{lower}$$

$$c_2 (x - x_0)(x - x_1) + c_1 (x - x_0) + c_0$$

$$= c_2 x^2 + \text{lower}$$

$$\left.\begin{array}{l} (c_2 x^2 + \text{lower})'' = \\ (c_2 2x + \text{lower})' \\ 2c_2 + 0 \end{array}\right\} \begin{array}{l} f'' \\ \| \\ f''(x) \end{array}$$

$g''(\eta) = 0$

$(f-p)''(\eta) = 0$

$$\underbrace{f''(\eta)}_{f''(\eta)} = \underbrace{p''(\eta)}_{2c_2}$$

Done!

So

$p(x) = f[x_0] + f[x_0, x_1](x - x_0)$

$\quad + f[x_0, x_1, x_2](x - x_0)(x - x_1)$

Slick formula for $f[x_0, x_1, x_2]$

Trick: Lagrange formula ---

for $p(x)$:

$$y_0 \underbrace{\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}}_{\text{poly deg 2 s.t.}}$$

$x_0 \mapsto 1$

$x_1 \mapsto 0$

$x_2 \mapsto 0$

$$+ \; y_1 \; \frac{\text{etc.}}{\text{etc.}}$$

$$+ \; y_2 \; \frac{\text{etc.}}{\text{etc.}}$$

$$\frac{(x-x_1)(x-x_2)}{\text{whatever}} = \frac{(x^2 + \text{lower})}{\text{whatever}}$$

So x^2 coeff in

$$\boxed{y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}}$$

is $x^2 \dfrac{y_0}{(x_0-x_1)(x_0-x_2)}$

so do the same for y_1, y_2

terms

$p(x) = c_2 x^2 + \text{lower}$

fits $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))$

OR $(x_0, y_0 \quad\quad y_1 \quad\quad y_2$

Highest is

$$c_2 = \frac{y_0}{(x_0-x_1)(x_0-x_2)} + \frac{y_1}{(x_1-x_0)(x_1-x_2)}$$

$$+ \frac{y_2}{(x_2-x_0)(x_2-x_1)}$$

$\left(\begin{array}{l}\text{OR} \\ y_i \to f(x_i)\end{array}\right)$

Recall

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_1, x_0] = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f[x_0, x_1]$$

Claim: $f[x_0, x_1, x_2]$ also doesn't depend on the order --

$$c_2 = \sum_{i=0}^{2} \frac{y_i}{\underbrace{(x_i - x_0) \cdots (x_i - x_n)}_{\text{remove } (x_i - x_i)}}$$

Reason 2 that

$$f[x_0, x_1, x_2] = f[x_0, x_2, x_1]$$

$$= f[x_1, x_2, x_0] = \cdots$$

We are finding unique deg 2 poly $p(x)$ s.t.

$$\left\{ \begin{array}{l} p(x_0) = y_0 = f(x_0) \\ p(x_1) = y_1 = f(x_1) \\ p(x_2) = y_2 = f(x_2) \end{array} \right\}$$

same if you write

$$p(x_1) = f(x_1)$$
$$p(x_2) = f(x_2)$$
$$p(x_0) = f(x_0)$$

General rem:

monomial:

$$p(x) = \tilde{c}_0 + \tilde{c}_1 x + \tilde{c}_2 x^2$$

div-diff

$$p(x) = c_0 + c_1 (x - x_0)$$

$$+ c_2 (x - x_0)(x - x_1)$$

Rem:

look at $1, x, x^2$

or

$1, \; x - x_0, \; (x - x_0)(x - x_1)$

the span of either ~~set~~ of functions is the same.

$x_0 = x_1 = 1$

claim: $1, x, x^2$

or

$1, (x-1), (x-1)^2$

$$\left\{ \alpha_0 \cdot 1 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 \;\middle|\; \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

or

$$\left\{ c_0 \cdot 1 + c_1 (x-1) + c_2 (x-2) \;\middle|\; c_0, c_1, c_2 \in \mathbb{R} \right\}$$

you get all poly's deg ≤ 2 ...

Why!

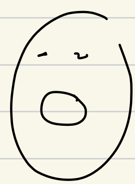
$$\alpha_0 + \alpha_1 x + \alpha_2 x^2$$

or

$$c_0 1 + c_1 (x-1) + c_2 (x-1)^2$$

$$= c_0 \cdot 1 + c_1 (x-1)$$

$$+ c_2 (x^2 - 2x + 1)$$



$$= 1 (c_0 - c_1 + \dots)$$

!

$$+ x$$

$$+ \dots$$

???