CPSC 303, March 1, 2024

- Divided differences:

$$p(x) = \underbrace{f[x_0]}_{\text{constant}} + \boxed{\underbrace{(x-x_0)}_{\text{poly}}}\, \underbrace{f[x_0,x_1]}_{\text{constant}} +$$

$$\cdots + \boxed{\underbrace{(x-x_0)\cdots(x-x_{n-1})}_{\text{poly}}}\, \underbrace{f[x_0,\ldots,x_n]}_{\text{const}}$$

Satisfies $p(x_i) = f(x_i)$, $i = 0,\ldots,n$.

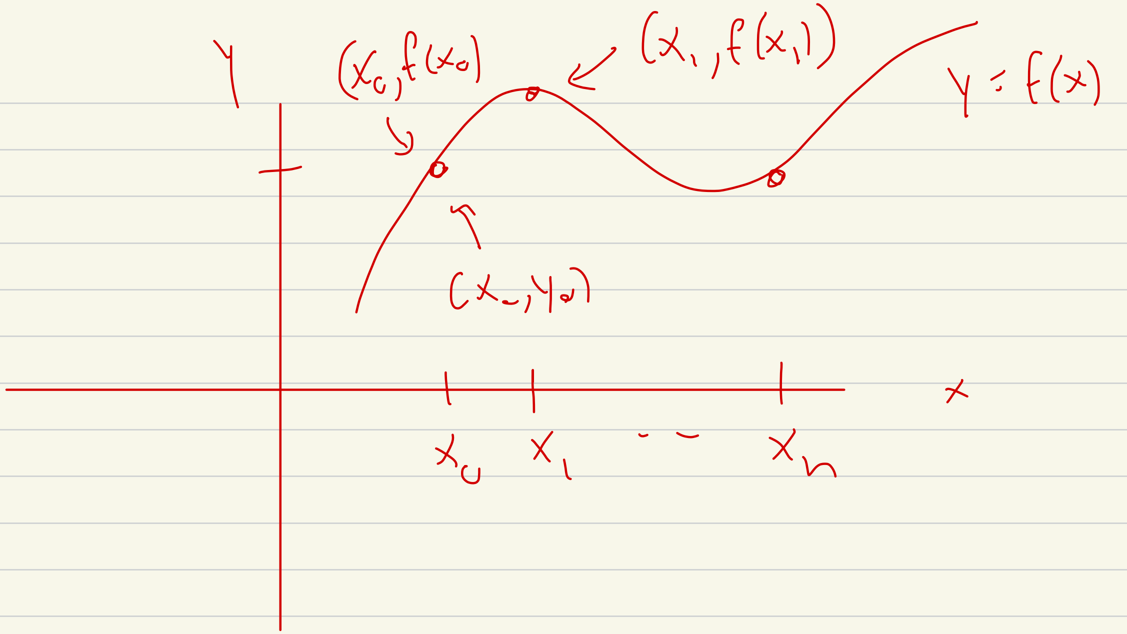
a <u>vast</u> generalization of Taylor's theorem

Note: 10.2, 10.3:

$$p(x_i) = y_i, \quad i = 0, \ldots, n$$

data points: $(x_0, y_0), \ldots, (x_n, y_n)$

10.4:



Instead of $\boxed{y_i = p(x_i)}$, $i = 0, \dots, n$

We fix $f : \mathbb{R} \to \mathbb{R}$, look for

p s.t. $\boxed{f(x_i) = p(x_i),}$ $i = 0, \dots, n$

Symbols $f[x_0]$, $f[x_0, x_1]$,

..., $f[x_0, x_1, \dots, x_n]$ } Divided differences

Midterm covers until

Feb 26 M

Feb 28 W

Moncmial Inter
Lagrange "
Conditicn Numbers

~~March 1~~
~~F~~

Divided diff

Final

Midterm F, March 15

Hamework 7 due March 7

Idea!

Say you have

$$y_i = p(x_i), \qquad i = 0, \dots, n$$

$$x_0 < x_1 < \dots < x_n$$

y_i really drawn from $f = f(x)$

so

$$y_i = f(x_i) \overset{\text{model}}{=} p(x_i)$$

$$\underbrace{p = p(x) = c_0 + c_1 x + \dots + c_n x^2}_{\text{Done the work to find } c_0, \dots, c_n}$$

Say you have

$$x_0 < x_1 < \dots < x_n < x_{n+1}$$

add $\quad y_{n+1} = f(x_{n+1})$

and we want poly q deg $n+1$,

and

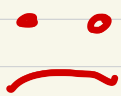
$$q(x_0) = y_0 = f(x_0)$$

$$\vdots$$

$$q(x_n) = y_n = f(x_n)$$

and

$$q(x_{n+1}) = y_{n+1} = f(x_{n+1})$$



We have

$$y_i = f(x_i) = p(x_i) \quad i=0,\dots,n$$

p deg $\leq n$.

Trick to find $q(x)$, using our knowledge of $p(x)$ s.t.

$$f(x_i) \text{ or } y_i = q(x_i) \quad i=0,\dots,n$$

and also

$$f(x_{n+1}) = y_{n+1} = q(x_{n+1})$$

$(\deg(q) \leq n+1$, unique$)$

q:

- Maybe Taylor series
- Lagrange multipliers ... idea!

$$\underbrace{(x-x_0)(x-x_1)\cdots(x-x_n)}_{\deg\, n+1}$$

$$= x^{n+1} + \text{lower}$$

So ...

$$q(x) = p(x) + \tilde{c}\,(x-x_0)\cdots(x-x_n)$$

Whatever $\tilde{c} \in \mathbb{R}$,

$$q(x_0) = p(x_0) + \tilde{c}(x_0 - x_0)(x_0 - x_1) \cdots$$

$$= p(x_0)$$

and similarly

$$q(x_1) = p(x_1) + 0$$

$$\vdots$$

$$q(x_n) = p(x_n) + 0$$

$$\cdots < x_n < x_{n+1}$$

Try!

$q(x_{n+1}) =$ y_{n+1} or $f(x_{n+1})$ ---

$$q(x_{n+1}) = p(x_{n+1}) + \hat{c}\,\boxed{(x_{n+1}-x_0)(x_{n+1}-x_1)\cdots(x_{n+1}-x_n)}$$

$\|$

y_{n+1} or

$f(x_{n+1})$

non-zero

So

$$\tilde{c} = \frac{y_{n+1} - p(x_{n+1})}{(x_{n+1}-x_0)\cdots(x_{n+1}-x_n)}$$

$$* \quad \boxed{\tilde{c} = \frac{f(x_{n+1}) - p(x_{n+1})}{(x_{n+1}-x_0)\cdots(x_{n+1}-x_n)}}$$

symbol: $\boxed{f[x_0, x_1, \ldots, x_{n+1}]}$

Def: If $f: \mathbb{R} \to \mathbb{R}$, and

$x_0 < \dots < x_{n+1}$ reals,

then we define

$f[x_0, \dots, x_{n+1}]$ to be

the unique $\tilde{c} \in \mathbb{R}$ s.t.

if p is the unique degree $\leq n$

s.t. $p(x_0) = f(x_0), \dots, p(x_n) = f(x_n)$

then $\tilde{c}$ is given by $(*)$

First: usual definition!

$$f[x_0] \stackrel{\text{def}}{\equiv} f(x_0)$$

If you want $q(x) = c_0$

to satisfy $q(x_0) = f(x_0)$,

then $q(x_0) = c_0 = f(x_0)$

Next

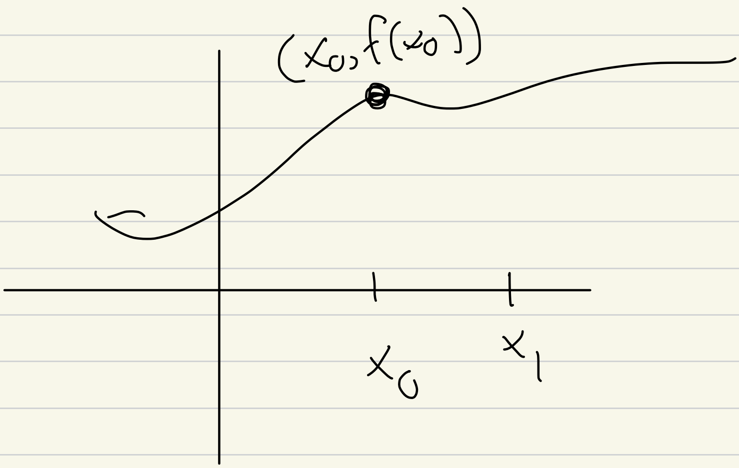
$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_1, x_0] = \frac{f(x_0) - f(x_1)}{x_0 - x_1} =$$

Now $p(x) = c_0$ s.t.

$$p(x_0) = f(x_0) = c_0,$$

$p(x) = f(x_0)$ satisfies $p(x_0) = f(x_0)$



$p(x) =$ constant function $f(x_0)$

add x_1:

$$q(x) = p(x) + \tilde{c}(x - x_0)$$

$$\underbrace{q(x_1)}_{\text{want } f(x_1)} = \underbrace{p(x_1)}_{f(x_0)} + \tilde{c}\,(x_1 - x_0)$$

$$\tilde{c} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Thm: For $x_0 < x_1$, $f: \mathbb{R} \to \mathbb{R}$

$$q(x) = f(x_0) + \tilde{c}\,(x - x_0)$$

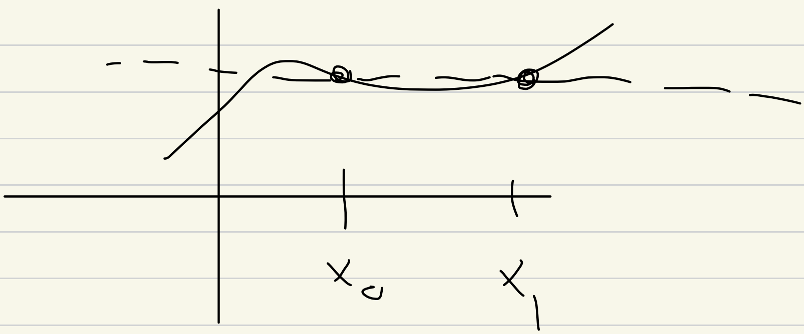
$$f[x_0, x_1] = \left\{ \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right\}$$

Then

$$\overset{f(x_0)}{\downarrow}$$

$$q(x) = f[x_0] + f[x_0, x_1](x - x_0)$$

is the line between



Mean Value Thm: If f is differentiable, then for some $x_0 < \xi < x_1$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi)$$

Next case: we have

$$\underbrace{f[x_0]}_{\tilde{c}_0} + \underbrace{f[x_0, x_1]}_{\tilde{c}_1}(x - x_0)$$

$$\tilde{c}_0 + \tilde{c}_1 \quad (x - x_0)$$

now embellish

to go through $(x_2, f(x_2))$

(x_2, y_2)

$$f[x_0] + f[x_0, x_1](x - x_0)$$

$$+ \tilde{c}_2 (x - x_0)(x - x_1)$$

and $\tilde{c}_2 =$

Thm: $\tilde{c}_2$, whatever it is

$\left(f[x_0, x_1, x_2] \right)$ equals

$\frac{1}{2} f''(\xi)$ . . .

Wow!

to be continued