CPSC 303, Feb 28, 2024

– Monic interpolation (Section 10.2)

versus Lagrange interpolation (Section 10.3)

– Divided differences (Sections 10.4-10.7)

See article:

CPSC 303: Remarks on Divided Differences

(likely to be revised to include more

comments on 10.4-10.7)

Likely look at the following...

Monore interpolation!

Data $(x_0, y_0), \ldots (x_n, y_n)$

Want

$$p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

s.t.

$$p(x_i) = y_i \qquad i = 0, \ldots, n$$

10.2 Vandermonde matrix

$$\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ & & \vdots & \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

What could possibly go wrong?

Say: $(x_0, y_0), (x_1, y_1)$

Say --

$$x_0 = 2, \qquad x_1 = 2 + 10^{-5}$$

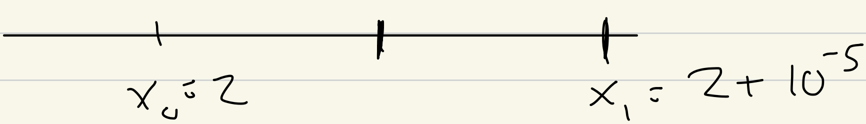
you just want

$$p(x) = c_0 + c_1 x,$$

find c_0, c_1, compute

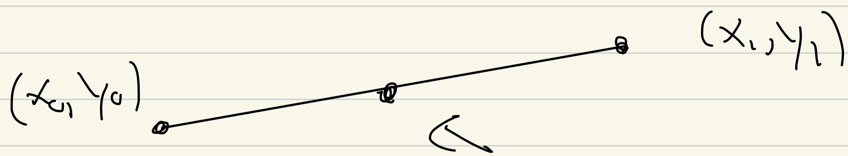
$$p\left(2 + \left(10^{-5}\right)\frac{1}{2}\right)$$

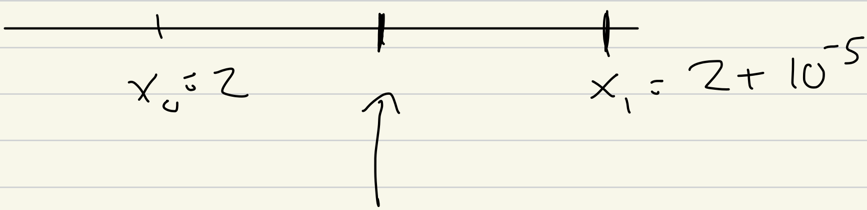
$$x_{mid} = 2 + 10^{-5} \cdot \frac{1}{2}$$



$$\begin{bmatrix} 1 & 2 \\ 1 & 2+10^{-5} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$p\left(2 + 10^{-5} \cdot \frac{1}{2}\right) = \frac{y_0 + y_1}{2}$$





$$x_{mid} = 2 + 10^{-5} \cdot \frac{1}{2}$$

(1) Numerically: Solve

$$\begin{bmatrix} 1 & 2 \\ 1 & 2+10^{-5} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$A\vec{c} = \vec{y}, \qquad A = \begin{bmatrix} 1 & 2 \\ 1 & 2+10^{-5} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\frac{1}{\underbrace{(2+10^{-5})-2}_{10^{5}}} \begin{bmatrix} 2+10^{-5} & -2 \\ -1 & 1 \end{bmatrix}$$

roughly

$$\left\| \begin{bmatrix} 1 & 2 \\ 1 & 2+10^{-5} \end{bmatrix} \right\|_\infty$$

$$= \max(1+2,\ 1+2+10^{-5}) \simeq 3+10^{-5}$$

$$\left\| \begin{bmatrix} 1 & 2 \\ 1 & 2+10^{-5} \end{bmatrix}^{-1} \right\|_\infty$$

$$= \frac{1}{\det} (4+10^{-5})$$

$$= 10^5 (4+10^{-5})$$

But consider Lagrange interpolation!

Say fitting $(x_0, \quad)$, $(x_1 \quad)$, $(x_2 \quad)$

Note! (fixed x_0, x_1, x_2)

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{\boxed{(x_0 - x_1)(x_0 - x_2)}}$$

constant

satisfies

$$L_0(x_1) = \frac{(x_1 - x_1) \text{ etc.}}{\text{etc.}} = 0$$

$$L_0(x_2) = 0$$

$$L_0(x_0) = \frac{(x_0 - x_1)(x_0 - x_2)}{(x_0 - x_1)(x_0 - x_2)} = 1$$

$p(x_0) = y_0, \quad p(x_1) = y_1, \quad p(x_2) = y_2$

$L_0 =$ poly deg 2 s.t.

$$L_0(x_0) = 1, \quad L_0(x_1) = L_0(x_2) = 0$$

and similarly for L_1, L_2, then

$$p(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

Magic:

$$L_0(x_0) = 1, \quad L_0(x_1) = 0, \quad L_0(x_2) = 0$$

$$L_1(x_0) = 0$$

$$L_2(x_0) = 0$$

$$\boxed{\begin{aligned} L_0(x_0) &= 1 \\ L_1(x_0) &= 0 \\ L_2(x_0) &= 0 \end{aligned}}$$

$$\Big(y_0 L_0(x) + y_1 L_1(x) + y_2 L(x) \Big)\Bigg|_{x \to x_0}$$

poly deg ≤ 2

$$= y_0 \, 1 + y_1 \cdot 0 + y_2 \cdot 0$$

$$= y_0$$

Linear case of Lagrange interpolation:

$$(x_0, y_0), \quad (x_1, y_1)$$

$$p(x) = y_0 \underbrace{\frac{(x-x_1)}{(x_0-x_1)}}_{\substack{\text{at } x = x_0 \to 1 \\ x = x_1 \to 0}} + y_1 \frac{(x-x_0)}{(x_1-x_0)}$$

What happens! $x_0 = 2, \quad x_1 = 2+10^{-5}$

want $p(x_{mid}) = p\left(2 + 10^{-5}\left(\frac{1}{2}\right)\right)$

Formula

$$p(x_{mid}) =$$

$$y_0\left(\frac{(x_{mid} - x_1)}{(x_0 - x_1)}\right) + y_1\left(\frac{(x_{mid} - x_0)}{(x_1 - x_0)}\right)$$

$$\frac{\left(\left((2 + 10^{-5} \cdot \tfrac{1}{2})\right) - \left((2+10^{-5})\right)\right)}{\left(2 - \left((2+10^{-5})\right)\right)}$$

2.000

$$2{,}1 = 2 + 10^{-1}$$

$$2{,}01 = 2 + 10^{-2}$$

$$2{,}00001 = 2 + 10^{-5}$$

$$2{,}000005 = 2 + 10^{-5} \cdot \frac{1}{2}$$

$$\underbrace{2.000005}_{\substack{\text{at least} \\ 10^{-16} \\ \text{rel prec.}}} - \underbrace{2.00001}_{\substack{\text{at least} \\ 10^{-16} \\ \text{rec prec}}}$$

$$= -\underbrace{.000005}_{\frac{1}{2} \cdot 10^{-5}} \pm 2 \cdot 10^{-16}$$

The point:

$$\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

and $\quad x_0 \to x_n$ close

$x_0 = 2$

$x_1 = 2+\varepsilon \qquad\qquad (\varepsilon$ small$)$

$x_2 = 2+2\varepsilon$

$\vdots$

$x \longrightarrow 2 + \left(\frac{1}{2}\right)\varepsilon$

This gives one situation where Lagrange interpolation loses less precision than monic interpolation.

Section 10.4 — 10.7 + beyond

Newton's Divided Differences...

Class ends
