CPSC 303   Feb 26, 2024

- Review:
  - Relative Error in Double Precision
  - " " & Condition Number

$$\kappa_p(A) \stackrel{\text{def}}{=} \|A\|_p \|A^{-1}\|_p$$

- Compare

  10.2 Monomial Interpolation

  10.3 Lagrange "

This week:

  10.4 Divided Differences

Basic Problem: CPSC 303 $\neq$ CPSC 302

Mention! Say you have sci. notation, base 10, remembering 4 digits

$1,000 \times 10^{71}$

$1,001 \times 10^{71}$

,

,

$9,999 \times 10^{71}$

$1,000 \times 10^{72}$

,

,

,

true value: 3.217569×10^{71}

sci notation to 4 places: 3.218×10^{71}

rel error:

$$\frac{|(\text{true value}) - (\text{sci notation})|}{|\text{true value}|}$$

$$= \frac{|3.217569 - 3.218|}{|3.218|}$$

$$< \frac{|0.0005|}{|3.218|}$$

max size:

$$\frac{\left|\frac{1}{2}\cdot 10^{-3}\right|}{|1{,}000|} = \frac{1}{2}\cdot 10^{-3}$$

rel error as high as

$$\underbrace{0{,}001}_{10^{-3}} \cdot \frac{1}{2}$$

$$10^{-3}\cdot\frac{1}{2}$$

Error in double precision for a standard number:

$$\pm\ 1.\underbrace{b_1 \ldots b_{52}}_{52} \cdot 2^{m}$$

$$(-1022 \leq m \leq 1023)$$

rel error max

$$2^{-52} \cdot \frac{1}{2} \simeq 2^{-53}$$

$$\boxed{\approx 1.1 \times 10^{-16}}$$

(In ODE's, this error compounds)
(In recurrences, " " ")

Say: Solving

$$A \vec{x}_{true} = \vec{b}_{true}$$

really $\vec{b}_{true} \rightsquigarrow \vec{b}_{observed}$

$$= \vec{b}_{true} + \vec{b}_{error}$$

$$A \vec{x}_{observed} = \underbrace{\vec{b}_{observed}}_{\neq \vec{b}_{true}}$$

true value of A

$$\vec{x}_{observed} = A^{-1}(\vec{b}_{observed})$$

Question: $1 \le p \le \infty$, using $\| \, \|_p$

Rel Error in $\vec{b}$:

$$\left(\begin{matrix}\text{Rel Error} \\ \text{in } \vec{b}\end{matrix}\right) = \frac{\| \vec{b}_{true} - \vec{b}_{observed} \|_p}{\| \vec{b}_{true} \|_p}$$

$$\frac{\| \vec{x}_{true} - \vec{x}_{observed} \|}{\| \vec{x}_{true} \|} \le C \, \frac{\| \vec{b}_{true} - \vec{b}_{observed} \|}{\| b_{true} \|}$$

(1) What is max C?

(2) When is this max attained?

Last week: $1 \leq p \leq \infty$

Worst C is

$$K_p(A) = \|A\|_p \, \|A^{-1}\|_p$$

... let's just believe this ...

Tricks!

$$A \vec{x}_{true} = \vec{b}_{true}$$

$$A \vec{x}_{observed} = \vec{b}_{observed}$$

$$A(\vec{x}_{true} - \vec{x}_{observed}) = \vec{b}_{error} \stackrel{def}{=} \vec{b}_{true} - \vec{b}_{obs}$$

$$A \, \vec{x}_{error} = \vec{b}_{error}$$

So

Rel Error in $\vec{x}$ Rel err in $\vec{b}$

$$\frac{\|\vec{x}_{error}\|_p}{\|\vec{x}_{true}\|_p} \le C \frac{\|\vec{b}_{error}\|_p}{\|\vec{b}_{true}\|_p}$$

???

$$A\vec{x}_{error} = \vec{b}_{error}$$

!!!

so $\vec{x}_{error} = A^{-1}\vec{b}_{error}$

Step ①:

What is max C_1 constant s.t.

$$\boxed{\|\vec{x}_{error}\|_p \leq C_1 \|\vec{b}_{error}\|_p}$$

② What is max C_2

$$\boxed{\frac{1}{\|\vec{x}_{true}\|_p} \leq C_2 \frac{1}{\|\vec{b}_{true}\|_p}}$$

where:

$\vec{b}_{error}$ is $\boxed{\text{anything}}$

$\vec{b}_{true}$ is $\boxed{\text{anything}}$

$$\vec{x}_{error} = A^{-1} \vec{b}_{error},$$

so

$$\|\vec{x}_{error}\|_p = \|A^{-1}\vec{b}_{error}\|_p$$

$$\leq \|A^{-1}\|_p \, \|\vec{b}_{error}\|_p$$

Also, you can have $\vec{x}_{error}, \vec{b}_{error}$ non-zero and

$$\|A^{-1}\vec{b}_{error}\|_p = \|A^{-1}\|_p \, \|\vec{b}_{error}\|_p$$

Since

$$\|M\|_p \overset{def}{=} \max_{\vec{x} \neq 0} \frac{\|M\vec{x}\|_p}{\|\vec{x}\|_p}$$

Remark: To find such a pair $\vec{b}_{error}$, $\vec{x}_{error} = A^{-1}\vec{b}_{error}$ easy for $\| \ \|_\infty$

$$\rightsquigarrow \left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\|_\infty$$

$$= \max\left(|a|+|b|, |c|+|d| \right)$$

Step 1:

$$\|\vec{x}_{error}\|_p \leq C \|\vec{b}_{error}\|_p$$

where C can be as low as $\|A^{-1}\|_p$

Step 2: Want C_2 s.t.

$$\frac{1}{\|\vec{x}_{true}\|_p} \le C_2 \frac{1}{\|\vec{b}_{true}\|_p}$$

(smallest possible C_2)

i.e. smallest C_2 s.t.

$$\|\vec{b}_{true}\|_p \le C_2 \|\vec{x}_{true}\|_p$$

$$\downarrow$$

$$\|\vec{b}_{true}\|_p = \|A\vec{x}_{true}\|_p$$

$$\le \|A\|_p \|\vec{x}_{true}\|_p$$

So smallest C_2 is

$$\|A\|_p,$$

$$\|\vec{b}_{true}\|_p \leq \|A\|_p \|\vec{x}_{true}\|_p$$

is attained with equality when

$$\|A\vec{x}_{true}\| = \|A\|_p \|\vec{x}_{true}\|_p.$$

best $C_1 = \|A^{-1}\|_p$

$C_2 = \|A\|_p$

hence

$$\frac{\|\vec{x}_{error}\|_p}{\|\vec{x}_{true}\|_p} \leq K(A) \frac{\|\vec{b}_{error}\|_p}{\|\vec{b}_{true}\|_p}$$

where

$$K(A) = \|A\|_p \, \|A^{-1}\|_p$$