CPSC 303, Feb 14, 2024

Condition Numbers and

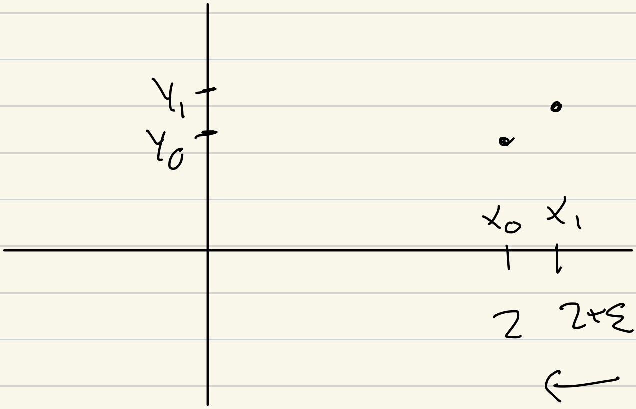
$$\begin{bmatrix} 1 & 2 \\ 1 & 2+\varepsilon \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$(x_0 = 2,\ x_1 = 2+\varepsilon)$ and similar problems.

Handout:

CPSC 303: What the Condition Number Does and Doesn't Tell Us

[A&G] §4.2, §5.8



$$\text{①}\quad c_0 + c_1 2 = y_0$$

$$\text{②}\quad c_0 + c_1(2+\varepsilon) = y_1$$

$$\text{②} - \text{①}\qquad c_1 \varepsilon = y_1 - y_0$$

$$c_1 = \frac{y_1 - y_0}{\varepsilon}$$

Why measurable sense is

$$\begin{bmatrix} 1 & 2 \\ 1 & 2+\varepsilon \end{bmatrix} \text{ bad ?}$$

Ans: It has a high condition number.

Norms of $\mathbb{R}^n$

$$\vec{x} = (x_1, \ldots, x_n)$$

$$\|\vec{x}\| = \|\vec{x}\|_2 \overset{\text{usual}}{=} \sqrt{x_1^2 + \ldots + x_n^2}$$

$$= \sqrt{\vec{x} \cdot \vec{x}}$$

$$\|\vec{x}\|_p = \left(|x_1|^p + \dots + |x_n|^p \right)^{1/p}$$

$$1 \le p \le \infty$$

$$\|\vec{x}\|_\infty = \lim_{p \to \infty} \|\vec{x}\|_p$$

$$\max_{i=1,\dots,n} |x_i|^p \le \underbrace{|x_1|^p + \dots + |x_n|^p} \le \left(\max_{i=1..n} |x_i|^p \right) n$$

$$\max |x_i| \le \left(\qquad \right)^{1/p} \le \left(\max |x_i| \right) n^{1/p}$$

So $p \to \infty$ $\quad \|\vec{x}\|_p \to \underbrace{\max_i (|x_i|)}_{\|\vec{x}\|_\infty \text{ or } \|\vec{x}\|_{max}}$

Norm on $\mathbb{R}^n$: $\vec{x} \to \underbrace{\|\vec{x}\|}_{\text{non-negative}}$

s.t.

1. $\|\vec{x}\| \geq 0$ with equality iff $\vec{x} = 0$

2. $\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|, \quad \alpha \in \mathbb{R}$

3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|, \quad \vec{x}, \vec{y} \in \mathbb{R}^n$

Is it obvious that

$$\|\vec{x}\|_p$$

then

$$\|\vec{x}+\vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$$

1 for $1 \leq p \leq \infty$

2 but not for $0 < p < 1$

Rem: Mostly $p = 1, 2, \infty$

$$\|\vec{x}\|_1 = |x_1| + \ldots + |x_n|$$

Let A i $m \times n$ matrix,

$\vec{x} \in \mathbb{R}^n$, $A\vec{x} \in \mathbb{R}^m$

Define $\|A\|_p$ to be

① $\displaystyle \max_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_p}{\|\vec{x}\|_p}$

OR

② $\displaystyle \max_{\|\vec{x}\|_p = 1} \|A\vec{x}\|_p$

OR

③ Smallest $C \geq 0$ s.t. $\|A\vec{x}\|_p \leq C \|\vec{x}\|_p$

E.g.

$$A = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{bmatrix}$$

$$\|A\vec{x}\|_p = \left\| \begin{bmatrix} d_1 x_1 \\ \vdots \\ d_n x_n \end{bmatrix} \right\|_p$$

$$\leq \left(\max_{i=1,\dots,n} |d_i| \right) \left\| \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\|_p$$

$$\begin{bmatrix} 2 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \end{bmatrix}$$

$$\left\| \begin{bmatrix} 0 \\ -7 \end{bmatrix} \right\|_p = 7 \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|_p$$

So

$$\left\| \begin{bmatrix} 2 & 0 \\ 0 & -7 \end{bmatrix} \right\|_p = 7$$

$=$

$$\left\| \begin{bmatrix} a \end{bmatrix} \right\|_p = |a| \qquad a \in \mathbb{R}$$

$$[a] \quad |x|$$

$$\left\|\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right\|_p = |a| + |b|$$

all $1 \le p \le \infty$

$a, b > 0$

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\left\|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\|_2 =$$

Turns out

$$\|A\|_2 = \sqrt{\text{largest eigenvalue of } AA^T}$$

$$= \text{largest singular value of } A$$

Handout

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 9 \end{bmatrix} \dashrightarrow ☹$$

$$\|A\| = \max_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2}$$

$$= \max_{\vec{x} \neq 0} \sqrt{\frac{\|A\vec{x}\|_2^2}{\|\vec{x}\|_2^2}}$$

$$= \sqrt{\max_{\vec{x} \neq 0} \frac{(A\vec{x}) \cdot (A\vec{x})}{\vec{x} \cdot \vec{x}}}$$

$$(A\vec{x}) \cdot (A\vec{x}) = (A\vec{x})^T (A\vec{x})$$

$$= \vec{x}^T A^T A \vec{x} = \vec{x} \cdot (A^T A \vec{x})$$

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 48 \\ 48 & 90 \end{bmatrix} \dots$$

Thm: If $A = A^T$, A is symmetric

$$\|A\|_2 = \max_{1 \le i \le n} |\lambda_i|$$

$$\left\|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\|_\infty$$

$$= \max\left(|a|+|b|,\ |c|+|d|\right)$$

$$\overbrace{\left(\max|a_{ij}|\right)}^{\text{Max Entry in abs value}} \le \|A\|_p \le n\left(\max|a_{ij}|\right)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{mj} & & a_{mn} \end{bmatrix}$$

$$\left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\|_1 = \max\left(|a|+|c|, |b|+|d| \right)$$

$\|A\|_\infty$ = max over all rows of the sum of absolute values in the row

$\|A\|_1 =$

Say A is $n \times n$, and you're solving

$A\vec{x} = \vec{b}$ (want)

actually $A\hat{\vec{x}} = \hat{\vec{b}}$ (are doing)

$\hat{\vec{b}}$ is your approximation to $\vec{b}$

$$\hat{\vec{b}} = \vec{b} + \vec{b}_{error}$$

<u>Theorem</u>: Relative error in

$\vec{x}$ (when you actually find

$\hat{\vec{x}}$) is

$$\leq K(A) \left(\begin{array}{l} \text{relative error} \\ \text{in } \vec{b} \text{ when} \\ \text{actually measure} \\ \quad \hat{\vec{b}} \end{array} \right)$$

where

$$\boxed{\kappa(A) = \|A\|_p \, \|A^{-1}\|_p}$$

and relative error:

$$\text{RelError}_p(\vec{b}, \hat{\vec{b}}) = \frac{\|\vec{b} - \hat{\vec{b}}\|_p}{\|\vec{b}\|}$$

$$\text{"} \quad (\vec{x}, \hat{\vec{x}}) = \ldots$$

$\varepsilon < 1$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} = A$$

$$\|A\|_\infty = 1$$

$$\|A^{-1}\|_\infty = \left\| \begin{bmatrix} 1 & 0 \\ 0 & 1/\varepsilon \end{bmatrix} \right\|$$

$$= 1/\varepsilon$$

$$K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

$$= 1/\varepsilon$$

$\varepsilon \to 0$ (bad)

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} x_\varepsilon \\ x_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$= x_0 = 1,$

$x_1 = 1/\varepsilon$

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1,001 \end{bmatrix}$$

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{bmatrix} 1 \\ 1.001/\varepsilon \end{bmatrix}$$