CPSC 303, Feb 12, 2024

Today:

- §10.2 as general method

- §10.2 $(x_0, y_0), (x_1, y_1), \ldots$

  what happens "as $x_1 \to x_0$"

"Monomial Interpolation"

Real Point:

§10.2 [A&G] "What could possibly go wrong?"

Ans: Condition number.

Recall:

$(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$

$n+1$ points:

Model:

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$$

Find $c_0, c_1, \ldots, c_n$ s.t.

$$p(x_i) = y_i \quad \text{for } i = 0, \ldots, n$$

Linear system for $c_0, c_1, \dots, c_n$

$$c_0 + c_1 x_0 + c_2 x_0^2 + \dots + c_n x_0^n = y_0$$

$$\vdots$$

$$c_0 + c_1 x_n + c_2 x_n^2 + \dots + c_n x_n^n = y_n$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \dots + & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

General Method ...

(x_0, y_0), (x_1, y_1), (x_2, y_2)

Model:

$$y_i = f(x_i)$$

$$= \boxed{c_0 \left(e^{e^{1/x_i}}\right)} + \boxed{c_1 \sin(\cos(x_i^3))}$$

$$+ \boxed{c_2 x_i^5}$$

(usually: poly's, sin, cos, e^x, $\ln x$)

Still

$$f(x) = \underbrace{c_0 g_0(x) + c_1 g_1(x) + c_2 g(x)}^{\text{linear curve fitting}}$$

$$f(x_0) = y_0$$

$$c_0 g_0(x_0) + c_1 g_1(x_0) + c_2 g(x_0) = y_0$$

$$\vdots$$

$$\underbrace{\begin{bmatrix} g_0(x_0) & g_1(x_0) & g_2(x_0) \\ g_0(x_1) & g_1(x_1) & g_2(x_1) \\ g_0(x_2) & g_1(x_2) & g_2(x_2) \end{bmatrix}} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

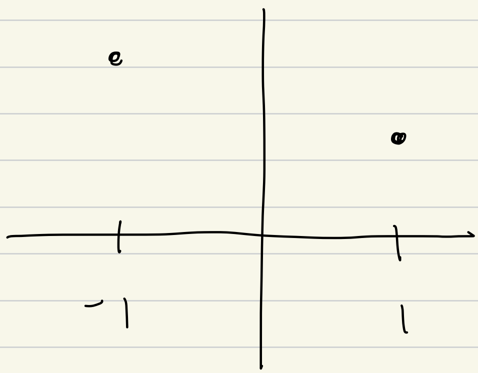
- is this matrix invertible?

- does the solution, assuming there is a unique, have some alternate forms

Rem! $f(x) = c_0 + c_1 x^2$

(x_0, y_0), (x_1, y_1)

-1 1



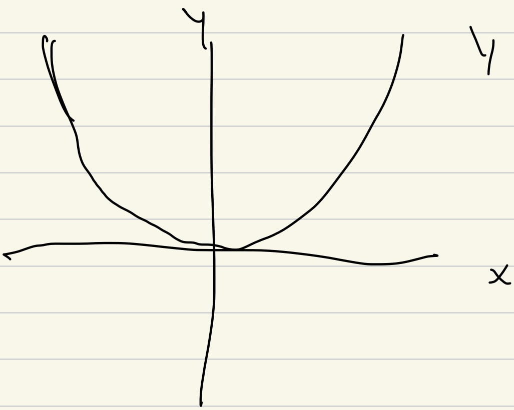
We want

$$y_0 = c_0 + c_1 (-1)^2$$

$$y_1 = c_0 + c_1 (1)^2$$

$$\begin{bmatrix} 1 & (-1)^2 \\ 1 & 1^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$



$y = x^2$

Try $f(x) = c_0 + c_1 x^2$

$$(x_0, y_0) \qquad (x_1, y_1)$$
$$\uparrow \qquad\qquad \uparrow$$
$$1 \qquad\qquad 2$$

$$\begin{bmatrix} 1 & (1)^2 \\ 1 & (2)^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

(similarly $\sin(x)$, $\sin(2x)$, ...
$\cos(x)$, $\cos(2x)$, ...)

Polynomial Interpolation:

Q: What if $x_1 \to x_0$,

e.g. x_0 fixed, $x_1 = x_0 + \varepsilon$

ε small (maybe $\varepsilon \to 0$)

$=$

(x_0, y_0), $(x_1 = x_0 + \varepsilon, y_1)$

$p(x) = c_0 + c_1 x$

$x_0 = 3$:

$$\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 1 & 3+\varepsilon \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$\Big\downarrow \; \varepsilon \to 0$$

$$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

$$\det = 1\cdot(3+\varepsilon) - 1\cdot 3 = \varepsilon$$

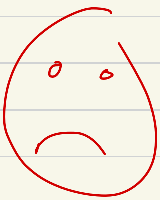
$$c_0 + 3c_1 = y_0$$

$$c_0 + (3+\varepsilon)c_1 = y_1$$

$$c_0 + 3c_1 = y_0$$

$$\varepsilon c_1 = y_1 - y_0$$

$$c_1 = \frac{y_1 - y_0}{\varepsilon}$$



$$c_0 = y_0 - 3c_1 = y_0 - 3\left(\frac{y_1 - y_0}{\varepsilon}\right)$$

If $(3, y_0)$, $(3+\varepsilon, y_1)$

coming from measurements

maybe

$(3, y_0)$ is really $(3, y_0(1 \pm 0.01))$

$(1 \pm 1\%)$

$(3+\varepsilon, y_1)$ " " $(3, y_1(1 \pm 0.01))$

As $\varepsilon \to 0$ (say $y_0, y_1 > 0$)

$$C_1 = \frac{y_1 - y_0}{\varepsilon} \pm 1\% \frac{y_1 + y_0}{\varepsilon}$$

2 ways:

Imagine $y_0 = f(3)$

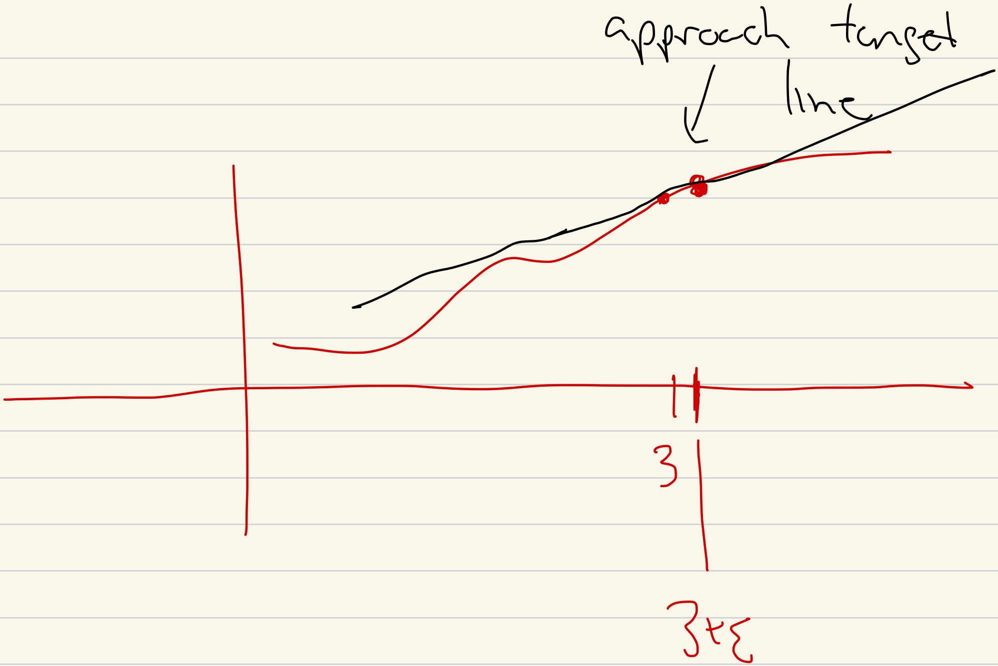
$y_1 = f(3+\varepsilon)$

for some function f.

What happens to

$$\begin{bmatrix} 1 & 3 \\ 1 & 3+\varepsilon \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} f(3) \\ f(3+\varepsilon) \end{bmatrix}$$

As $\varepsilon \to 0$ ...



$$C_1 = \frac{Y_q - Y_0}{\varepsilon} = \frac{f(3+\varepsilon) - f(3)}{\varepsilon}$$

As $\varepsilon \to 0$

$$C_1 = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = f'(3)$$

As $\varepsilon \to 0$ $\quad c_1 \to f'(3)$

$$c_0 + 3c_1 = f(3)$$

$$c_0 = f(3) - 3c_1$$

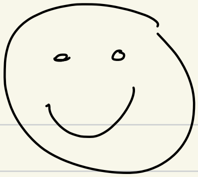
$$= f(3) - 3f'(3)$$

line to approximate

$$c_0 + c_1 x$$

$$= \left(f(3) - 3f'(3)\right) + f'(3)x$$

$$= f(3) + (x-3)f'(3)$$



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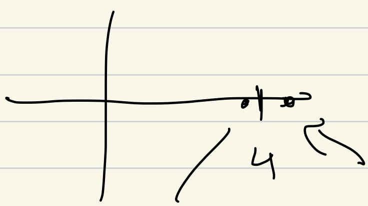
What about

$$(x_0, y_0), \quad (x_1, y_1), \quad (x_2, y_2)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$4 \qquad\quad 4+5h \qquad\quad 4-h$$

$h \to 0$ ... ...



best parabola to fit

$f(4+h)$

$$= f(4) + h f'(4) + \frac{h^2}{2} f''(4)$$

$$+ O(h^3)$$ (assuming f''' exists near 4, - -)

$f(x) =$

$$f(4) + (x-4) f'(4)$$

$$+ \frac{1}{2}(x-4)^2 f''(4) +$$

$$+ O((x-4)^3)$$

$$\begin{bmatrix} 1 & 4 & 4^2 \\ 1 & 4+5h & (4+5h)^2 \\ 1 & 4-h & (4-h)^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f(4) \\ f(4+5h) \\ f(4-h) \end{bmatrix}$$

$$\vdots$$

should get Taylor's thm

+ $f''(4)$ approx with

$f(4), f(4+.5h), f(4-h)$