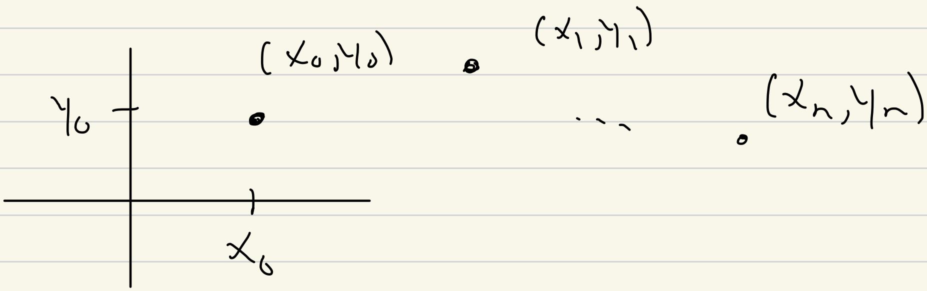
CPSC 303, Feb 9, 2024

- Group Homework 5
- Practice Homework 5

Start Ch 10 [A&G] Interpolation



<u>Thm:</u> Given $(x_0, y_0), \ldots, (x_n, y_n)$ there is unique poly $p(x)$ of deg $\le n$ s.t.

$p(x_i) = y_i$ for all $i = 0, \ldots, n$

<u>Pf 1:</u> [A&G] 10.2: Monomial Interpolation

<u>Pf 2</u>: [A&G] 10.3: Lagrange Interpolation

<u>Pf 3</u>: [A&G] 10.4: Divided Differences and Newton's form

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3 Proofs $\Rightarrow$ 3 "methods" or "algorithms," all with advantages & disadvantages.

HW #5:

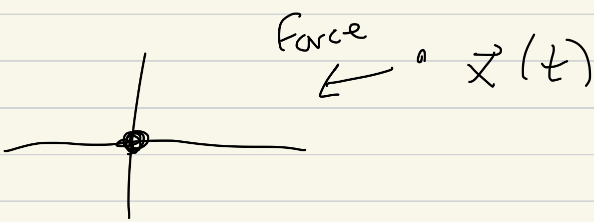
- Euler's Method
- Trapezoid Method

$\}$ HW #4

Give you some MATLAB code to solve central force problem:

$$m\ddot{\vec{x}}(t) = m\,u(\|x\|)\left(\frac{-\vec{x}}{\|\vec{x}\|_2}\right)$$

mass $\cdot$ acceleration $=$ Force



Newton's: $a(\|x\|) = \dfrac{1}{\|x\|_2^2}$

Energy: independent of time

$$\underbrace{\frac{1}{2} m \|\vec{v}\|_2^2}_{\text{kenetic enery}} - \underbrace{m\, U(\|x\|_2)}_{\text{potential energy}}$$

Where $U = U(r)$, $a(r)$,

$$r = \sqrt{x_1^2 + x_2^2} = \|x\|_2$$

$$\left(\vec{v} = \dot{\vec{x}}, \quad \vec{a} = \dot{\vec{v}} = \ddot{\vec{x}}\right)$$

m = mass

$=$

$$\ddot{\vec{x}} = g(t, \vec{x}, \dot{\vec{x}})$$

or just

$$g(\vec{x})$$

$$y = \begin{pmatrix} \dot{\vec{x}} \\ \vec{x} \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_1 \\ x_2 \end{pmatrix}$$

$$\vec{x} = (x_1, x_2)$$

$$\vec{x}(t) = (x_1(t), x_2(t))$$

$$\frac{d}{dt}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ x_1\\ x_2\end{pmatrix} = \begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} \\ \\ y_1\\ y_2\end{pmatrix}$$

functions of y_3, y_4

$$\begin{pmatrix} y_1 = \dot{x}_1\\ y_2 = \dot{x}_2\\ y_3 = x_1\\ y_4 = x_2\end{pmatrix}$$

$$\frac{d}{dt}\vec{y} = \frac{d}{dt}\begin{pmatrix}y_1\\ y_2\\ y_3\\ y_4\end{pmatrix} = f(\vec{y})$$

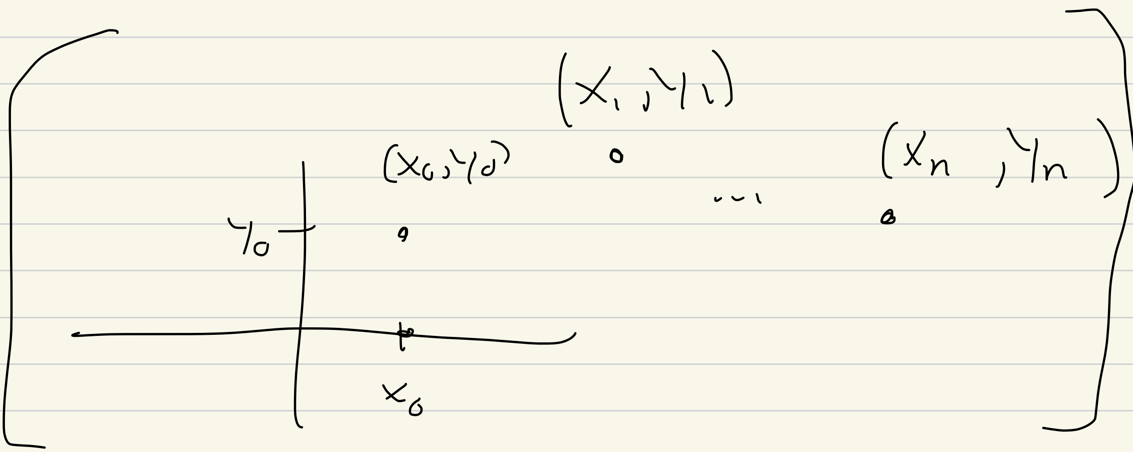
Ch 10, Section 2 [A & G]

Monomial Interpolation:

Thm: Given $n+1$ points in $\mathbb{R}^2$,

$(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$

s.t. $x_0 < x_1 < \ldots < x_n$



Then there is a unique polynomial

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_0$$

s.t.

$$\forall\, i = 0, \dots, n \qquad y_i = p(x_i).$$

Pf 1 : You want

$$y_0 = c_n x_0^n + c_{n-1} x_0^{n-1} + \dots + c_0$$

$$y_1 = c_n x_1^n + \dots + c_0$$

$$\vdots$$

$$y_n = c_n x_n^n + \dots + c_0$$

So want to find $c_0, \dots, c_n$ s.t.

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & & \vdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Claim: This matrix is invertible --- It's $(n+1) \times (n+1)$ a square matrix.

From linear alg (echelon form):

if you have a system $n+1$ equations $n+1$ variables, have unique solution iff the homogeneous form does!

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ \vdots & & & & \\ & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

So!

$p(x_1) = 0$

$p(x_2) = 0$

$\vdots$

$p(x_n) = 0$

So p poly degree $\leq n$, but

p has $n+1$ roots:

$x_0, x_1, \ldots, x_n$

$\Longrightarrow p(x) = 0$ (the zero poly)

i.e.

$c_0 = c_1 = c_2 = \cdots = c_n = 0$

"Vandermonde"

Class ends

$p(x)$ has $p(x_0) = 0$, then

$$p(x) = (x - x_0)\underbrace{q(x)}_{\text{a polynomial}}$$

$p(x_1) = 0 \quad x_1 \neq x_0$

then $q(x_1) = 0$

so $q(x) = (x - x_1) r(x)$

So

$$p(x) = (x - x_0)(x - x_1) \varphi(x)$$

$\vdots$

$$p(x) = (x - x_0)(x - x_1) \cdots (x - x_n) s(x)$$

$$(x^{n+1} + \text{lower} \qquad) s(x)$$

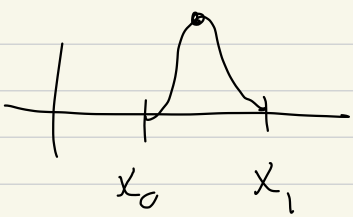
$p(x)$ has roots

$$x_0 < x_1 < x_2 - - \quad < x_n$$

Rolles thm:

$$p(\cancel{x}_0) = p(x_1) = 0, \Rightarrow x_0 < x_0' < x_1$$

s.t. $\quad p'(x_0') = 0$



p' has roots

$$x_0' < x_1 < x_1' < x_2 < x_2' - - < x_{n-1}' < x_n$$

p' has roots

$$x_0' < x_1' < \cdots < x_{n-1}'$$

n roots

p'' has $n-1$ roots

$\vdots$

$$\boxed{\begin{array}{l} p^{(n)} \text{ has } 1 \text{ root} \\ p(x) = c_n x^n + c_1 x^{n-1} + \cdots \end{array}}$$

$p^{(n)}(x) = c_n n! \Rightarrow c_n = 0$

$p^{(n-1)}$ has 2 roots!

$$p^{(n-1)} = \cancel{c_0\, n!\, x} + c_1 (n-1)!$$

$$\Rightarrow \quad c_1 = 0$$

$c_0 = c$, $c_1 = 0$, — — —

$$(S-2)^2(x_n) = 0$$

$$x_{n+2} - 4x_{n-1} + 4x_n = 0$$

Claim: Given x_0, x_1, there is a unique solution.

But we know

$$x_n = c_1 2^n + c_2 n 2^n$$

any c_1, c_2 work.

Note

won't work

$$x_n = c_1 2^n + c_2\, 200 \cdot 2^n$$

$$= (c_1 + c_2\, 200)\, 2^n$$

We have to check

$$c_1 2^0 + c_2 0 \cdot 2^0 = x_0$$

$$c_1 2^1 + c_2 1 \cdot 2^1 = x_1$$

there is a solution

$$\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

$c_1 = x_0$, $\quad 2c_1 + 2c_2 = x_1$

$$c_2 = \frac{x_1 - 2c_1}{2} = \frac{x_1}{2} - x_0$$

$$(c_1 + c_2 200)\, 2^0 = x_0$$

$$(c_1 + c_2 200)\, 2^1 = x_1$$

$$\begin{bmatrix} 1 & 200 \\ 2 & 400 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$