CPSC 303, Feb 5, 2024

- 3 Handouts!
  - Intro to ODE's
  - Recurrence Relations & Finite Precision
  - Normal & Subnormal Numbers

Appendices A, B, C

To now

Today: A bit more on 1st two handouts
- Start Normal & Subnormal Numbers

3-Term recurrence relations:

$$x_{n+2} = a\,x_{n+1} + b\,x_n$$

$a, b \in \mathbb{R}$, $\quad \ldots x_{-1}, x_0, x_1, \ldots$

$b \neq 0 \qquad \left(b = 0,\ a \neq 0 \quad \begin{matrix} x_{n+2} = a x_{n+1} \\ x_{n+1} = a x_n \end{matrix} \right)$

Analog of 2^{nd} order ODE:

$$F_{n+2} = F_{n+1} + F_n$$

$$\vec{Y}_{n+1} = A\,\vec{Y}_n$$

$$\vec{Y}_{n+1} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n \\ F_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

($F_n = 0$, $F_{n+1} - a F_n = 0$

kinda like $\underbrace{(F_{n+1} - F_n)}_{\text{kinda like } (\Delta f)_n} = (a-1) F_n$

(In terms of σ,

$\Delta = \sigma - 1$ so

$$(\Delta x)_n = (\sigma - 1)(x_n)$$
$$= x_{n+1} - x_n$$

)

Last time:

$$x_{n+2} - 4x_{n+1} + 4x_n = 0$$

$$(\sigma^2 - 4\sigma + 4)(x_n) = 0$$

Guess $x_n = r^n$ is a solution ...

...

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0, \quad r = 2, 2$$

$x_n = 2^n$ is a solution,

$x_n = c_1 2^n$ " " "

Think of it ... $(r-2)(r-2-\varepsilon) = 0$

$\varepsilon > 0$:

$$(\sigma - 2)(\sigma - 2 - \varepsilon)(x_n) = 0$$

Then:

$$x_n = c_1 2^n + c_2 (2+\varepsilon)^n$$

Fix $\varepsilon > 0$, have $x_0 = 3$, $x_1 = -40$,

$x_2 = x_2(\varepsilon)$, x_3 - - -

$$x_{21}(\varepsilon) = (\qquad) x_0 + (\qquad) x_1$$

given x_0, x_1

Taking $\varepsilon \to 0$, $x_{21}(0) = \_\_\_ x_0 + \_\_\_ x_1$

General solution to

$$x_{n+2} - 4x_{n+1} + 4x_n = 0$$

if

$$x_n = c_1 2^n + c_2 \underbrace{n 2^n}_{\text{like ODE's...}}$$

Does this work?

$$(n+2) 2^{n+2} - 4(n+1) \cdot 2^{n+1}$$

$$- 4 \, 2^n \overset{??}{=} 0$$

Try this

$$(\sigma-2)(\sigma-2)\,(n2^n) \stackrel{?}{=} 0$$

$$(\sigma-2)(n2^n)$$

$$= (n+1)2^{n+1} - 2(n2^n)$$

$$= 2^n\big((n+1)2 - 2n\big)$$

$$= 2^n(2)$$

And $(\sigma-2)(2^n\cdot\text{const}) =$

$$\text{const}\left(2^{n+1} - 2\cdot 2^n\right) = 0$$

$=$

What about

$$(\sigma-2)^4 (x_n) = 0 \quad - \text{—}$$

$$x_n = c_1 2^n + c_2 n 2^n$$

$$+ c_3 n^2 2^n + c_4 n^3 2^n$$

$$? \qquad\qquad ?$$

$$(\sigma-2)\left(p(n)\, 2^n\right)$$

$= \quad 2^n($ poly of lower degree

$$(\sigma-2)\left(n^k 2^n\right)$$

$$= (n+1)^k 2^{n+1} - n^k 2^n \cdot 2$$

$$= 2^{n+1}\left(\underbrace{(n+1)}\,^k - n^k\right)$$

$$= 2^{n+1}\left(k n^{k-1} + \text{lower}\right)$$

think of x^k, kx^{k-1} ---

like a derivative ---

Now:

$$(\sigma - r_1)^{m_1} \cdots (\sigma - r_k)^{m_k}(x_n) = 0$$

we get

$$x_n = p_1(n) r_1^n + \cdots + p_k(n) r_k^n$$

where $\deg(p_i) \leq m_k - 1$

---

e.g.

$$(\sigma - 2)^2 (\sigma - 5)(x_n) = 0$$

$$x_n = c_1 2^n + c_2 n 2^n + c_3 5^n$$

$$x_{n+2} - 4x_{n+1} + 4x_n = 0$$

$$\vec{Y}_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

$$\vec{Y}_{n+1} = \begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

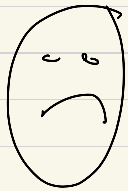
So if we look for eigenvalues:

$$\det\left(\lambda \cdot I - \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} \lambda - 4 & 4 \\ -1 & \lambda \end{bmatrix} = (\lambda - 4)\lambda + 4 = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\lambda = 2, 2$$



What if

$$A = S \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} S^{-1}$$

$$= 2 \underbrace{S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} S^{-1}$$

$$= 2 \; S \; S^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Look at $\lambda I - \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$

$\lambda = 2$:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$A\vec{v} = 2\vec{v}$, then $\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \vec{v} = 0$

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \left(\cdot \begin{bmatrix} r \\ 1 \end{bmatrix} \right)$$

So $\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$ can't be

diagonalized ---

Since $2I - \begin{bmatrix} \; \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}}_{N} \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{??}$$

Can $N \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

but $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$??

We say N is <u>nilpotent</u>

if $N^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for some

$k \geq 1$.

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} = 2 \cdot I - \begin{bmatrix} \text{matrix} \\ \text{of} \\ \text{interest} \end{bmatrix}$$

$= N =$ nilpoint

$$\begin{bmatrix} \text{matrix} \\ \text{of} \\ \text{interest} \end{bmatrix} = 2 \cdot I - N$$

$$\begin{bmatrix} \quad \end{bmatrix}^k = (2 \cdot I - N)^k =$$

$$(2I)^k + \binom{k}{1}(2I)^{k-1}(-N)$$

$$+ \binom{k}{2}(2I)^{k-2}\cancelto{0}{N^2}$$

$$+ \cdots$$

$$= 2^k I - 2^{k-1}\binom{k}{1} N$$

$$+ 0$$

Note: Correction to notes was made on Feb 12, indicated in red, thanks to Anon. Beaker on piazza.