CPSC 303, Feb 2, 2024

Gray Larson, The Far Side:

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What you dog Rover hears when you speak:

blah blah blah Rover blah blah

blah blah blah blah blah blah

blah Rover blah blah blah blah !

blah blah blah blah blah blah

blah blah blah blah blah blah

blah blah blah Rover blah blah

blah blah blah blah blah blah

blah blah blah blah blah blah

What does your cat Mittens
hear when you speak?

blah blah blah blah blah blah
blah blah blah blah blah blah
blah blah blah blah blah blah
blah blah blah blah blah blah
blah blah blah blah blah blah
blah blah blah blah blah blah
blah blah blah blah blah blah
blah blah blah blah blah blah

Thm: Let $y' = f(t,y)$, $y(t_0) = y_0$,

blah blah ... blah

-- exists --- continuous ---

unique -- Lipschitz continuous --

- — - — -

<u>Proof</u> (Appendix A)

blah blah blah blah blah blah

blah blah blah blah blah blah

blah blah blah blah blah blah

blah blah blah blah blah blah

blah blah blah blah blah blah

blah blah blah blah blah blah

blah blah blah blah blah blah

blah blah blah blah blah blah

Thm: Let $f = f(n, x)$,

$f: \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ and consider

$$X_{n+1} = f(n, X_n) \quad \text{for all } n \geq n_0$$

and

$$X_{n_0} = X_0$$

Then, there exists a unique solution

$$X_{n_0}, X_{n_0+1}, X_{n_0+2}, \ldots$$

$$X : \{n_0, n_0+1, n_0+2, \ldots\} \to \mathbb{R}$$

Pf: $X_{n_0+1} = f(n_0, X_{n_0})$, $X_{n_0+2} = f(n_0+1, X_{n_0+1})$

$\ldots$

Rem: By construct,

$$x_{n_0-1} \overset{??}{\leftarrow} x_{n_0}, x_{n_0+1}, \ldots$$

It's not clear that

$$x_{n_0} = f(n_0-1, x_{n_0-1})$$

can even be solved...

⋮

ODE's $\longleftrightarrow$ Recurrences

Often ODE's are limit (Euler's method) of recurrences (depend on $h>0$)

Reverse engineer

$$(r-3)(r-2) \longleftrightarrow x_n = c_1 2^n + c_2 3^n$$

$$r^2 - 5r + 6 = 0$$

$$\Big\}$$

$$(\sigma^2 - 5\sigma + 6)(x_n) = 0$$

$$\Big\}$$

$$x_{n+2} - 5x_{n+1} + 6x_n = 0$$

So say want to solve:

$$x_{n+2} - 5x_{n+1} + 6x_n = 0$$

Guess ... maybe ...

$x_n = r^n$ will be a

solution for some r ...

x_{-2}	x_{-1}	x_0	x_1	x_2	...
$1/r^2$	$1/r$	1	r	r^2	

let's try:

$$x_{n+2} - 5x_{n+1} + 6x_n = 0$$

$$r^{n+2} - 5r^{n+1} + 6r^n = 0$$

(if $r \neq 0$) holds for all $n \in \mathbb{Z}$

$\Longleftrightarrow$ $= \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0$$

So $x_n = 2^n$ work, 3^n work

,

$x_0 = 1, \quad x_1 = 2, \quad x_2 = 4$ hopefully...

$$x_{n+2} \stackrel{?}{=} 5x_{n+1} - 6x_n \text{ ??}$$

$$4 \stackrel{\checkmark}{=} \underbrace{5 \cdot 2 - 6 \cdot 1}_{4} \qquad (n=0)$$

$$8 \text{ vs } 5 \cdot 4 - 6 \cdot 2 \qquad (n=1)$$

Similarly for $x_n = 3^n$, so

$x_n = c_1 2^n + c_2 3^n$ work

Thm: For $x_{n+2} = 5x_{n+1} - 6x_n$

for any x_0, x_1 there is

a unique solution

Pf 1: $x_2 = $ function x_1, x_0

$x_3 = $ " x_2, x_1

$\vdots$

and

$$6x_n = 5x_{n+1} - x_{n+2}$$

$$x_n = (5x_{n+1} - x_{n+2})/6$$

$x_{-1} = $ function(x_0, x_1), – ~

Pf 2:

We know $x_n = c_1 2^n + c_2 3^n$ is a solution. So find c_1, c_2 that work...

$$c_1 2^0 + c_2 3^0 = x_0$$

$$c_1 2^1 + c_2 3^1 = x_1$$

or x_0, x_1 given, solve

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

Similarly if we had

$$x_n = c_1 2^n + c_2 3^n + c_3 7^n$$

via

$$(\sigma - 2)(\sigma - 3)(\sigma - 7)(x_n) = 0$$

to solve for c_1, c_2, c_3

given x_0, x_1, x_2

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 7 \\ 2^2 & 3^2 & 7^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$\left(x_{n+3} = \ldots x_{n+2} \ldots x_{n+1} \ldots x_n \right)$$

Back to

$$x_{n+2} = 5x_{n+1} - 6x_n$$

$$x_{n+2} - 5x_{n+1} + 6x_n = 0$$

So ....

let $\vec{z}_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$, $n \in \mathbb{Z}$

$$\vec{z}_{n+1} = \begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

$$= \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} \vec{z}_n$$

We guessed $X_n = r^n$ for $(r = 2, 3)$ is a solution

$$\vec{z}_n = \begin{bmatrix} X_{n+1} \\ X_n \end{bmatrix} = \begin{bmatrix} r^{n+1} \\ r^n \end{bmatrix}$$

$$\vec{z}_{n+1} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \vec{z}_n$$

$$\underbrace{\begin{bmatrix} r^{n+2} \\ r^{n+1} \end{bmatrix}}_{r \begin{bmatrix} r^{n+1} \\ r^n \end{bmatrix}} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r^{n+1} \\ r^n \end{bmatrix}$$

$$r \begin{bmatrix} r^{n+1} \\ r^n \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r^{n+1} \\ r^n \end{bmatrix}$$

$\Leftrightarrow$ $(r \neq 0)$

$$r \begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ 1 \end{bmatrix}$$

So ... if our guess is correct, then

$$\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ 1 \end{bmatrix} = r \begin{bmatrix} r \\ 1 \end{bmatrix}$$

eigenvector, eigenvalue

So if $r=2$, $r=3$ work...

$$\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

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Compute eigenvalues of $\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$:

$$\det\left(t \cdot I - \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \right)$$

$$= \det \begin{pmatrix} \lambda - 5 & 6 \\ -1 & \lambda - 0 \end{pmatrix}$$

$$= (\lambda - 5)\lambda - (-1)(6)$$

$$= \lambda^2 - 5\lambda + 6 \quad !!$$

$$X_{n+2} = 4X_{n+1} - 4X_n$$

$$X_{n+2} - 4X_{n+1} + 4X_n = 0$$

$$(\sigma^2 - 4\sigma + 4)(X_n) = 0$$

$$(\sigma - 2)^2 (X_n) = 0$$

$$(\sigma-2)^2 \, (x_n) = 0$$

guess

$x_n = 2^n$ will work

but ... what else ...

there's no r^n for $r \neq 2$ ...

$$r^2 - 4r + 4 = 0$$

$(r-2)^2 = 0$ ... $r = 2, 2$

... what might work ... ?