CPSC 303, Jan 31, 2024

- Recurrences:

  General algorithm:

$$F_{n+2} - F_{n-1} - F_n = 0$$

$$(\sigma^2 - \sigma - 1)(F_n) = 0$$

solve $r^2 - r - 1 = 0$

$$\left(\sigma - \frac{1+\sqrt{5}}{2}\right)\left(\sigma - \frac{1-\sqrt{5}}{2}\right)(F_n) = 0$$

- ODEs with constant coefficients:

$$y'' - y' - y = 0$$

$$\left(\frac{d}{dt} - \frac{1+\sqrt{5}}{2}\right)\left(\frac{d}{dt} - \frac{1-\sqrt{5}}{2}\right) y = 0$$

- Multiple roots

Last time:

$$(*) \quad F_{n+2} - F_{n+1} - F_n = 0 \qquad \forall n$$

given $\begin{matrix} F_1 \\ F_0 \end{matrix} \rightsquigarrow F_2, F_3, \ldots$

also go backwards! $\begin{matrix} F_1 \\ F_0 \end{matrix} \rightsquigarrow F_{-1}, F_{-2}, \ldots$

=

we write $(*)$ as

$$\left((\sigma^2 - \sigma - 1)(F) \right)_n = 0$$

— ?

σ = shift operator:

$$(\sigma F)_n = F_{n+1}$$

think of

$$\begin{tikzcd}[column sep=small]
\big(\dots F_{-2} & F_{-1} & F_0 & F_1 & F_2 & \dots \big) \\
\dots & (\sigma F)_{-1} \arrow[ur] & (\sigma F)_0 \arrow[ur] & (\sigma F)_1 \arrow[ur] & , \dots \arrow[ur] &
\end{tikzcd}$$

so

$$\sigma : \begin{pmatrix} \text{sequences} \\ \text{indexed} \\ \text{over} \\ 0, 1, 2, \dots \end{pmatrix} \to \begin{pmatrix} \text{sequence} \\ \text{indexed} \\ \text{over} \\ 0, 1, 2, \dots \end{pmatrix}$$

$$\sigma : \begin{pmatrix} \text{sequences} \\ \text{indexed} \\ \text{over} \\ \dots, -2, -1, 0, 1, 2, \dots \end{pmatrix} \to \begin{pmatrix} \text{sequence} \\ \text{indexed} \\ \text{over} \\ -2, -1, 0, 1, 2, \dots \end{pmatrix}$$

$(\sigma^{-1} F)_n = F_{n-1}$ if bi-infinite sequence

If you accept this definition (or some other)

$$F_{n+2} - F_{n+1} - F_n = 0 \quad (\forall n)$$

$$\Updownarrow$$

$$\left(\sigma^2 F - \sigma F - F \right)_n = 0 \ (\forall n)$$

or

$$(\sigma^2 - \sigma - 1) F_{\bullet} = 0 \text{ sequence}$$

Recipe: $\Downarrow$

$$r^2 - r - 1 = 0$$

solve ...

get

$$r = \frac{1 \pm \sqrt{5}}{2}$$

$$(\sigma^2 - \sigma - 1) F = 0$$

$$\Updownarrow$$

$$\left(\sigma - \frac{1+\sqrt{5}}{2}\right)\left(\sigma - \frac{1-\sqrt{5}}{2}\right) F = 0$$

$$\Updownarrow$$

$$\left(\sigma - \frac{1-\sqrt{5}}{2}\right)\left(\sigma - \frac{1+\sqrt{5}}{2}\right) F = 0$$

Guess...

$$F_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$1, \sigma, \sigma^2, \sigma^3$ acts ~~on~~ functions

or

Id

$\{0, 1, 2, \dots\} \to \mathbb{R}$

e.g. $n \mapsto F_n$

$$p(x) = p_1(x)\, p_2(x) \quad \text{(polynomials)}$$

also have

$$p(\sigma) = p_1(\sigma)\, p_2(\sigma)$$

e.g. $(\sigma - 1)(\sigma - 2)$

$$= \sigma^2 - 3\sigma + 2 = (\sigma - 2)(\sigma - 1)$$

Look by comparison:

$$y' = Ay$$

general solution $y(t) = Ce^{At}$

i.e. $y(t) = Ce^{At}$, it satisfies

$$\frac{d}{dt} y = Ay$$

or

$$\left(\frac{d}{dt} - A\right) y = 0 .$$

Say you have

$$y'' - y' - y = 0$$

Recipe:

$$\left(\left(\frac{d}{dt}\right)^2 - \left(\frac{d}{dt}\right) - 1 \right) y = 0$$

$y = y(t)$.

$$\boxed{\left(\frac{d}{dt} - \frac{1+\sqrt{5}}{2} \right) \left(\frac{d}{dt} - \frac{1-\sqrt{5}}{2} \right) y = 0}$$

(we know, e.g.

$$\frac{d}{dt}\, 3 \,\frac{d}{dt}\, y = \frac{d}{dt}^2 (3y)$$

$$= 3\left(\left(\frac{d}{dt}\right)^2 y \right) \ldots \quad)$$

We know

$$\left(\begin{array}{l}\text{anything} \cdots \\ \text{taking} \\ c \to 0\end{array}\right)\left(\frac{d}{dt} - \frac{1-\sqrt{5}}{2}\right) y = 0$$

is satisfied by

$$y(t) = c_2 e^{\left(\frac{1-\sqrt{5}}{2}\right)t}$$

But --- same equation $y'' - y' - y = 0$

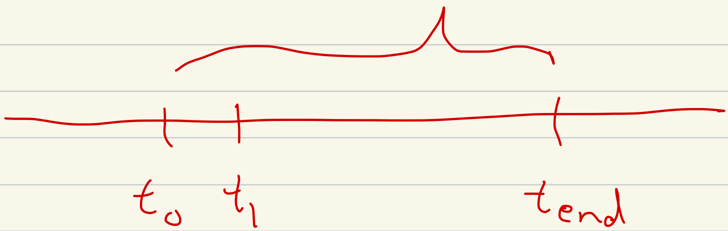
equivalent to

$$\underbrace{\left(\frac{d}{dt} - \frac{1-\sqrt{5}}{2}\right)}_{\text{ignore}}\left(\frac{d}{dt} - \frac{1+\sqrt{5}}{2}\right) y = 0$$

we get

$$y(t) = c_1 e^{\left(\frac{1+\sqrt{5}}{2}\right)t} + c_2 e^{\left(\frac{1-\sqrt{5}}{2}\right)t}$$

Recall: (t_1, y_1) (t_0, y_0) $\qquad t_{end} - t_0 = Nh$



approximate $y(t_{end})$ via Euler's method for $y' = Ay$,

then

$$y_1 = y_0 + hAy_0$$

$$= (1 + hA)y_0 \quad \ldots$$

$$y_N = (1 + hA)^N y_0, \quad \lim h \to 0$$

Solve

$$y'' - y' - y = e^{\sin(t)} + 20t^3$$

say we find

$$z(t) \text{ s.t.}$$

$$z'' - z' - z =$$

then if z is a solution,

and w solves the "homogeneous"

version $w'' - w' - w = 0$

then

$y(t) = z(t) + w(t)$ is another solution

Say $y = y(t)$

$$\left(\frac{d}{dt} - 2\right)^2 y = 0$$

then

$$y = c_1 e^{2t} + c_2 t e^{2t}$$

$$\left(\frac{d}{dt} - 2\right)\left(te^{2t}\right)$$

$$= \underbrace{\frac{d}{dt}\left(te^{2t}\right)} - 2te^{2t}$$

$$e^{2t} + t2e^{2t} - 2te^{2t} = e^{2t}$$

Or

$$\left(\frac{d}{dt} - 0\right)^2 y = 0$$

$$y'' = 0$$

$$y(t) = c_1 + c_2 t$$

we know $y(4) = c_1 e^{0 \cdot t} = c_1$

is a solution ...

$\vdots$

$$(\sigma - 0)^2 F = 0$$

$$\sigma^2 F = 0$$

$$f_0 \quad f_1 \quad f_2 \quad f_3 \quad \cdots$$

$$(\sigma^2 f)_0 \quad (\sigma^2 f)_1$$

solution $f_n = 0 \quad n \geq 2$

f_0, f_1 anything

$$(\sigma - 1)^2 F = 0$$

$$\sigma^2 F - 2\sigma F + F = 0$$

$$F_{n+2} - 2F_{n+1} + F_n = 0$$

If --

$$F_n = c_1 1^n + n 1^n c_2$$

$$= c_1 + c_2 n$$

works ...

Also --

given F_0, F_1

F_2 = determined

F_3 = ...

$\vdots$