CPSC 303, Jan 29, 2023

- e^{At}, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- Other reasons to write

$$A = S \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} S^{-1}$$

other than e^{At}, namely recurrences and A^n, $n = 0, 1, \ldots$

- Examples of

(1) diagonalizable A

(2) defective A (recurrences)
("not enough eigenvectors")

What is a 2×2 defective matrix? $n\times n$ "

Means (real matrixes) a matrix that is not diagonalizable over $\mathbb{C}$,

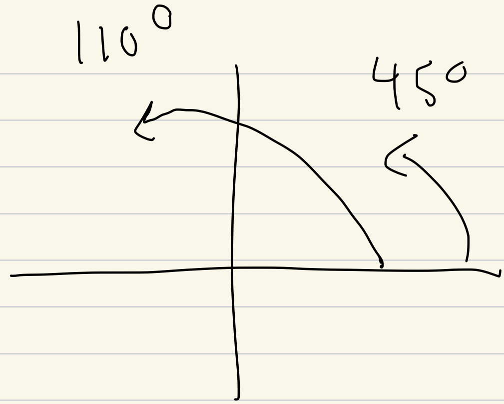
Diagonalizable:

$$A = S \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix} S^{-1}$$

hence

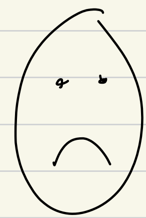
$$f(A) = S \begin{bmatrix} f(\lambda_1) & & 0\text{'s} \\ & f(\lambda_2) & \\ 0\text{'s} & & \ddots & \\ & & & f(\lambda_n) \end{bmatrix} S^{-1}$$

Rotation:



will have

- real entries
- complex (unit length) eigenvalues



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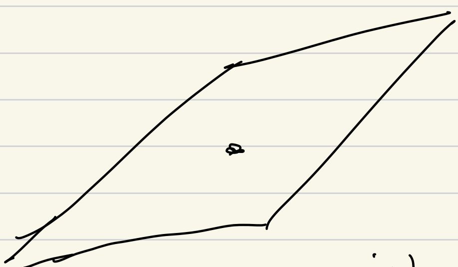
Projection onto 2-dim plane



you'll have eigenvalue 1 with multiplicity 2

Recurrence Relations & Finite Precision

Fibonacci numbers:

$$\ldots, -8, 5, -3, 2, \underset{F_{-2}}{-1}, \underset{F_{-1}}{1}, \underset{F_0}{0}, \underset{F_1}{1}, \underset{F_2}{1}, \underset{F_3}{2}, 3, 5, 8, 13, 21, 34, \ldots$$

$$F_{n+2} = F_{n+1} + F_n$$

or $$F_n = F_{n+2} - F_{n+1}$$

Also $F_1 = 1$, $F_2 = 1$ (initial conditions)

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} \begin{matrix} = & F_{n+1} + F_n \\ = & F_{n+1} \end{matrix}$$

(like $\dot{v} = -$
$\dot{x} = v$

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

$$A \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

$$\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

$$= A \cdot A \cdot \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

$$= \cdots$$

$$= A^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

Strong connection to

Euler's method, $\vec{y}\,' = A\vec{y}$.

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

How to diagonalize, at

2x2 matrix ---

Recipe: look for $\vec{v}$ s.t.

$$A\vec{v} = \lambda\vec{v} \qquad (\vec{v} \neq 0)$$

eigenvector

eigenvalue

$$A\vec{v} = \lambda\vec{v} = \lambda \cdot I \vec{v}$$

So

$$(A - \lambda I)\vec{v} = 0$$

So look for λ's s.t.

$A - \lambda I$:

1. has non-zero vector in its nullspace
2. det $= 0$
3. rank $< n$ (for $n \times n$)
4. not invertible
5. . . .

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(-\lambda) - (1)(1) = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

golden ratio and its "conjugate"

$$\lambda = \frac{1 \pm \sqrt{1 - -4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2} = 1.6\ldots,\ -.6\ldots$$

$\lambda = \frac{1+\sqrt{5}}{2}$; $\vec{v}$ s.t.,

$$\begin{bmatrix} \left(1 - \frac{1+\sqrt{5}}{2}\right) & (1) \\ (1) & \left(1 - \frac{1+\sqrt{5}}{2}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\left(1 - \frac{1+\sqrt{5}}{2}\right) v_1 + v_2 = 0$$

take $v_2 = 1$ . . . .

$$\left(A - I\left(\frac{1+\sqrt{5}}{2}\right)\right) \begin{bmatrix} ? \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} ? \\ \vdots \\ 1 \end{bmatrix} = \left(\frac{1+\sqrt{5}}{2}\right) \begin{bmatrix} ? \\ \vdots \\ 1 \end{bmatrix}$$

$=$

$$F_{n+2} = F_{n+1} + F_n \qquad \cancel{F_1 = 1,\ F_2 = 1}$$

now $F_0 =$ any

$F_1 =$ anything else

Say!

$F_0 = 1$, $F_1 =$

$F_2 =$, $F_3 =$. ~

Simpler recurrence

$$\cancel{F_{n+2} = F_{n+1} + F_n}$$

$$F_{n+1} = 10\, F_n$$

one-term recurrence

$$F_0 = 3, \quad F_1 = 3 \cdot 10$$

$$F_2 = (3 \cdot 10)\, 10 = 3 \cdot 10^2$$

$$F_3 = 3 \cdot 10^3 \ldots$$

$$F_n = 10^n F_0$$

$F_0 = 1$ or (anything)

is there an easy to write down solution:

$$F_n = \begin{pmatrix} \text{something} \\ \text{simple} \end{pmatrix} F_0$$

for $F_{n+2} = F_{n+1} + F_n$??

Ans:

$$F_n = \left(\frac{1+\sqrt{5}}{2}\right)^n F_0$$

and

$$F_n = \left(\frac{1-\sqrt{5}}{2}\right)^n F_0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda^2 \\ \lambda \end{bmatrix}$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$= \lambda \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$$

So!

one solution

$$F_n = C_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + C_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

If

$$\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = A \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = A \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

$$A\vec{v} = \lambda \vec{v}$$

$$A \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda^2 \\ \lambda \end{bmatrix}$$

Provided that: A expresses a recurrence, λ is an eigenvalue