CPSC 303, Jan 26, 2024

COMING SOON ...

- More ODE's!
  - Central Force:

$$m\ddot{\vec{x}} = -\left(\frac{\vec{x}}{|\vec{x}|}\right) g(|\vec{x}|)$$

Newton's Law: $g(r) = \frac{1}{r^2}$

  - n-body problem, celestial mech.

- Other common examples:

$$y' = y(1-y) \quad \text{(logistic)}$$

$$y'' = \sin(y) \quad \text{(pendulum)}$$

- (Edward) Lorenz attractor

- Time reversal of Newton's laws and a central force problem.

- Etc.

- Numerical issues! e.g. stiffness

END OF COMING ATTRACTIONS...

Higher Order ODE solvers

For now $\begin{cases} \text{Euler's Method} \\ \text{Explicit Trapezoidal} \end{cases}$

- More on e^{At}, norms, convergence, etc.
- Where do non-diagonalizable come from?

One place: recurrences

$$\boxed{y' = f(t,y), \quad y(t_0) = y_0}$$

$$y(t) - y(t_0) = \int_{s=t_0}^{s=t} y'(s)\, ds$$

$$y(t) - y_0$$

$$\boxed{y(t) - y_0 = \int_{s=t_0}^{s=t} f(s,y)\, ds}$$

"Integral form of

Rem: $f(s,y) = f(s)$

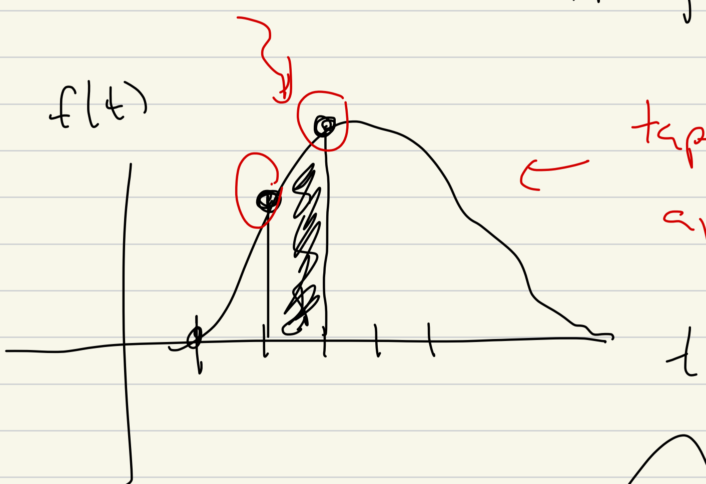
then:

$$\frac{dy}{dt} = y' = f(t), \quad y(t_0) = y_0$$

$$y(t) - y(t_0) = \int_{s=t_0}^{s=t} f(s)\, ds$$

$\int$

integration



trapezoidal approx

better than

rectangular approx

rectangle approx

$$y(t_0+h) = y(t_0) + h\,y'(t_0) + O(h^2)$$

$$\cdots \quad t_{-2} \quad t_{-1} \quad t_0 \quad t_1 \quad t_2 \quad t_3 \quad \cdots$$

$$t_i = t_0 + i h$$

Y_1 apprx $y(t_1)$

$$y(t_1) = y(t_0+h)$$

$$\approx y(t_0) + h\,y'(t_0) + O(h^2)$$

$$Y_{i+1} = Y_i + h\,f(t_0, Y_0)$$

So y_{i+1} approximates $y(t_{i+1})$

(uses y_i, approx $y(t_i)$) ...

To improve:

$$y(t+h) \approx y(t) + h$$ Better approx for y'

$$= y(t) + h\,\frac{y'(t) + y'(t+h)}{2}$$

...

Trapezoidal!

$$\frac{y(t+h) - y(t)}{2} \quad \text{to} \quad y'\left(t+\frac{h}{2}\right)$$

...

So first ...

$y(t_0)$, h

Euler: $y(t_1) = y(t_0+h)$

$$\approx y(t_0) + h\, y'(t_0)$$

$$= y(t_0) + h\, f(t_0, y_0)$$

$$y_1 \leftarrow y(t_0) + h f(t_0, y_0)$$

Say (Trapezoidal (Explicit) Scheme)

$$Y = y(t_0) + h\, f(t_0, y_0)$$

$$y(t_0+h) \approx y(t_0) + h\left(\frac{f(t_0,y_0) + f(t_1, Y)}{2}\right)$$

Question: What do we mean by

$$e^A = I + A + \frac{A^2}{2} + \dots \quad ?$$

=

If $\vec{x} \in \mathbb{R}^n$, then

$\|\vec{x}\|_2$ or $\|\vec{x}\|_{L^2}$ is

$$\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

So if $M \in \mathbb{R}^{n \times n}$, i.e. an $n \times n$

matrix, real entries

Then

$$\|M\|_{\substack{L^2 \\ \text{matrix}}} = \max_{\vec{x} \neq \vec{0}} \frac{\|M\vec{x}\|_2}{\|\vec{x}\|_2}$$

So $\|M\|_{\substack{L^2 \\ \text{metric}}}$ is the smallest

real B s.t.

$$\|M\vec{x}\|_{L^2} \leq \|\vec{x}\|_{L^2} B$$

E.g. $\vec{x} \in \mathbb{R}^n$, $n=1$, $\vec{x} = x_1 \in \mathbb{R}^1$

so $A \in \mathbb{R}^1$

$$A = [3]$$

$$\max \left(\frac{\| A\vec{x} \|_{L^2}}{\| \vec{x} \|_{L^2}} \right)$$

$$\vec{x} = (x_1), \qquad \| \vec{x} \|_{L^2} = \sqrt{x_1^2}$$

$$= |x_1|$$

$$\max_{x_1 \neq 0} \frac{|3x_1|}{|x_1|} = 3$$

So $A \in \mathbb{R}^{1\times 1}$, $A = [a]$

$$\|A\|_{L^2} = |a|$$

$$\|\vec{x}\|_{max} = \|\vec{x}\|_{\infty}$$

$$= \max\left(|x_1|, |x_2|, \ldots, |x_n|\right)$$

$$\|A\|_{L^\infty} = \max_{\vec{x} \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

Look at 2×2 matrices ...

Say

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\text{size}(A) = \max\left(|a_{11}|, |a_{12}|, |a_{21}|, |a_{22}|\right)$$

Claim:

$$\|A\|_\infty, \|A\|_2, \text{size}(A)$$

$=$

What is

$$I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots$$

What does convergence mean?

$$I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots + \frac{A^r}{r!} = r\text{-th term}$$

$$I + A + \frac{A^2}{2} + \dots + \frac{A^r}{r!} + \frac{A^{(r+1)}}{(r+1)!}$$

$$+ \dots + \frac{A^s}{s!} = s\text{-th term}$$

$$\underset{\substack{s, r \to \infty \\ s > r}}{\cancel{\text{sum}}\ \text{what}} \left(\frac{A^{r+1}}{(r+1)!} + \dots + \frac{A^s}{s!} \right)$$

$$\longrightarrow 0$$

meaning

$$\lim_{\substack{s,r\to\infty \\ s>r}} \left\| \left(\frac{A^{r+1}}{(r+1)!} + \dots + \frac{A^s}{s!} \right) \right\|_2 \to 0$$

$$\| \quad \|_\infty \to 0$$

$$\text{size} \left(\quad \right) \to 0$$

$$e^{At}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(so happens
$A^2 = I$)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 A = I A = A$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = A^2 = A^4 = A^6 \ldots$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A = A^3 = A^5 = \ldots$$

$$e^{At} =$$

$$I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \frac{(At)^4}{4!} + \cdots$$

$$\begin{bmatrix} 1 + \frac{t^2}{2} + \frac{t^4}{4!} + \cdots & t + \frac{t^3}{3!} + \frac{t^5}{5!} \\ & \end{bmatrix}$$

$$= \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}$$