CPSC 303, Jan 24, 2024

- Solutions to HW 2 and HW 1

Topics to finish

- Look at MATLAB a bit.
- Vandermonde matrices and Linear Algebra w/o Linear Algebra
- Exponentiation & Eigenpairs & Norms
- Higher Order versions of Euler's method
- More celestial mechanics

3 point schemes

$$\left\{\begin{array}{l} f(x_0+h) = \cdots \\ f(x_0) = \cdots \\ f(x_0-h) = \cdots \\ f(x_0+2h) = \cdots \\ f(x_0+\sqrt{2}\,h) = \cdots \\ f(x_0+rh) = \cdots \end{array}\right.$$

real number

often integers, often $\{0, 1, 2, -1, -2, \ldots\}$

$$c_0 \cdot f(x_0 + r_0 h) = c_0 \left(f(x_0) + r_0 h f'(x_0) + r_0^2 \frac{h^2}{2} f''(x_0) + r_0^3 \frac{h^3}{3!} f'''(x_0) + O(h^4) \right)$$

$$c_1 \cdot f(x_0 + r_1 h) = c_1 (\quad)$$

$$c_2 \cdot f(x_0 + r_2 h) = c_2 (\quad)$$

$$c_3 \cdot f(x_0 + r_3 h) = c_3 (\quad)$$

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sum to be

$$0 \cdot f(x_0) + 1 \; h f'(x_0) + 0 \; \frac{h^2}{2} f''(x_0) + 0 \; \frac{h^3}{3!} f'''(x_0)$$

$$(f(x_0)) \quad c_0 + c_1 + c_2 + c_3 = 0$$

$$(hf'(x_0)) \quad r_0 c_0 + r_1 c_1 + r_2 c_2 + r_3 c_3 = 1$$

$$f'': \quad r_0^2 c_0 + r_1^2 c_1 + r_2^2 c_2 + r_3^2 c_3 = 0$$

$$r_0^3 c_0 + r_1^3 c_1 + r_2^3 c_2 + r_3^3 c_3 = 0$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ r_0 & r_1 & r_2 & r_3 \\ r_0^2 & r_1^2 & r_2^2 & r_3^2 \\ r_0^3 & r_1^3 & r_2^3 & r_3^3 \end{bmatrix}}_{\text{Vandermonde}} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

f'

Looks like interpolation

Theorem: The above system has a unique solution.

Lemma: Say that $p : \mathbb{R} \to \mathbb{R}$, $p = p(x)$ is a polynomial and p has $n+1$ distict roots, $x_0, \ldots x_n$.

Then

① $p(x) = (x - x_0)(x - x_1) \ldots (x - x_n) \cdot q(x)$

where $q(x)$ is a polynomial.

② $p'(x)$ has n distinct roots

between x_0 and x_n

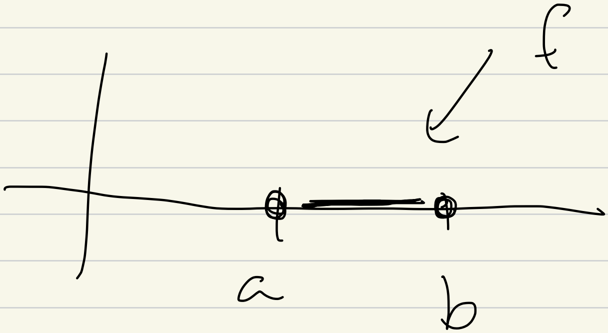
(assuming $x_0 < x_1 < \dots < x_n$)

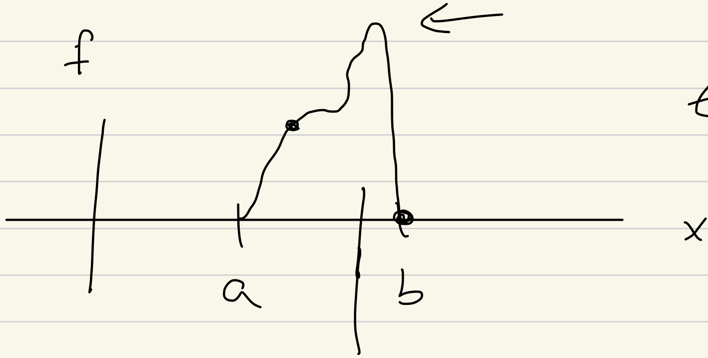
(3) $p''(x)$ has $n-1$ distinct roots

between x_0 and x_n

(4) $p'''(x)$ has $n-2$ distinct ...

Rolle's Thm: If $f \in C^0[a,b]$ (continuous on $[a,b]$) and $f(a)=f(b)=0$, f is differentiable on (a,b), then $f'(\xi)=0$ for some $a<\xi<b$.





$f > 0$ somewhere

max $x = \xi$

$f'(\xi) = 0$

Cor: If $p = p(x)$ is a poly of degree $\leq n$, $p(x) = c_0 + c_1 x + \dots + c_n x^n$, and p has $n+1$ distinct roots, then $p = 0$.

Rem! $p(x) = x^2+1$,

then p has no real roots

But $p'(x) = 2x$ has a real root.

Claim: The system

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has $c_1 = c_2 = \cdots = c_n = 0$ as its unique solution.

This just says

$$p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$$

satisfies

$$\left.\begin{aligned} p(x_0) &= 0 \\ p(x_1) &= 0 \\ &\vdots \\ p(x_n) &= 0 \end{aligned}\right\} \Rightarrow$$

p is a poly with $n+1$ distinct roots,

so

p has to be 0 poly

and $c_0 = c_1 = \dots = c_n = 0$

Homework!

$$e^{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t}$$

$$= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$e^{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} t}$$

$$= \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}$$