CPSC 303, Jan 22, 2024

We say A and B, two

$m \times m$ matrices, are <u>similar</u>

if for some invertible $m \times m$ matrix

S, we have

$$A = S B S^{-1}.$$

If so: $A^2 = S B S^{-1} S B S^{-1}$

$= S B^2 S^{-1}$

similarly $A^k = S B^k S^{-1}$

similarly

$$A^2 + 20 A^{30} = S \left(B^2 + 20 B^{30} \right) S^{-1}$$

So ...

$$e^{At} = ??$$

$$e^{At}, \quad A = \text{scalar} \ldots$$

$$e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$$

So ...

$$e^{A} = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \ldots$$

The point:

$$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(I + At + \frac{(At)^2}{2} + \frac{(At)^3}{3!} + \ldots\right)$$

$$= A + \frac{A^2 2t}{2} + \frac{A^3 3t^2}{3!} + \ldots$$

$$= A\left(I + At + \frac{(At)^2}{2!} + \cdots \right)$$

$$= A e^{At} = e^{At} A$$

So if you consider, some $m \times m$, A, some $t_0 \in \mathbb{R}$, $\vec{y_0} \in \mathbb{R}^m$

$$f(t) = e^{A(t-t_0)} \vec{y_0}$$

$$f'(t) = \left(e^{A(t-t_0)}\right)' (\vec{y_0}) + e^{A(t-t_0)} \left(\vec{y_0}'\right)$$

$$= A e^{A(t-t_0)} \vec{y_0} = A f(t)$$

$$A = S B S^{-1}$$

say $$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

then $$B^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

$$p(A) = S\, p(B)\, S^{-1}$$ p any poly

$$= S\, p\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\right) S^{-1}$$

$$= S \begin{bmatrix} p(\lambda_1) & 0 \\ 0 & p(\lambda_2) \end{bmatrix} S^{-1}$$

Similarly: If

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

any globally convergent power series

$f = f(x)$

$$f(A) = S \begin{bmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{bmatrix} S^{-1}$$

So $e^{At} = S \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} S^{-1}$

So!

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

eigenvalues: - -

If

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix} = (a+b) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a-b \\ b-a \end{bmatrix} = (a-b) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then

$$\underbrace{\begin{bmatrix} a & b \\ b & a \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{S} = \begin{bmatrix} a+b & a-b \\ a+b & b-a \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{S}\underbrace{\begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}}_{B}$$

So:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} = S\,B\,S^{-1}$$

$$= S\begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}S^{-1}$$

So

$$e^{\begin{bmatrix} a & b \\ b & a \end{bmatrix}} = S \begin{bmatrix} e^{a+b} & 0 \\ 0 & e^{a-b} \end{bmatrix} S^{-1}$$

$$=$$

Rem! Any matrix of the form:

$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, such as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

can be diagonalized in this way:

$$A = S \begin{pmatrix} \text{some} \\ \text{diagonal} \\ \text{matrix} \end{pmatrix} S^{-1}$$

Last time ... before ...

$$f(x_0) = f(x_0)$$

$$f(x_0+h) = f(x_0) + hf'(x_0) + \cdots$$

$$f(x_0+2h) = f(x_0) + 2hf'(x_0) + \cdots$$

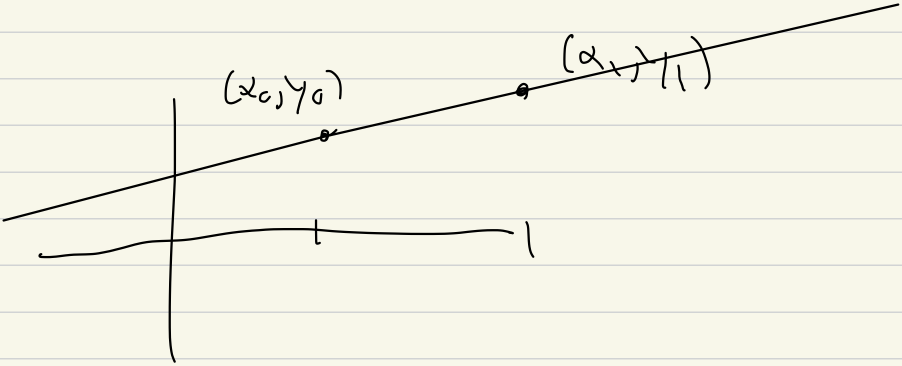
Created derivative schemes

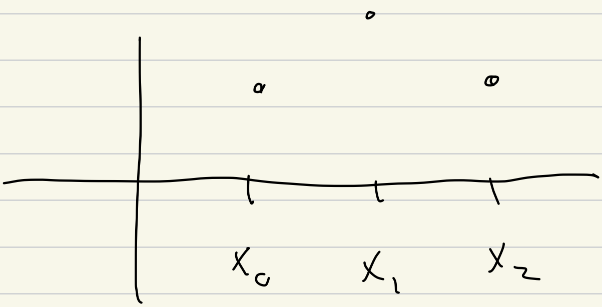
In [A&G], these are

stated in an ad hoc fashion...

For us ... Vandermonde matrices

So ... interpolation ...

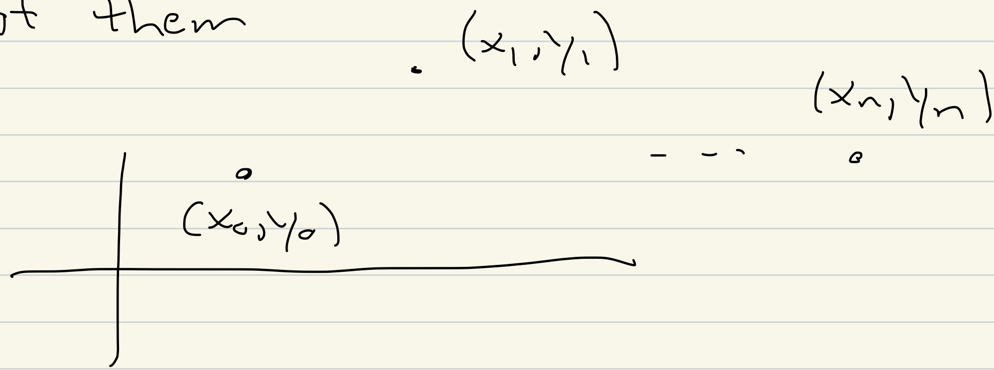




3 points, there's a unique parabola

Theorem: [Linear Algebra without Linear Algebra calculations...]

Say you have data points, $n+1$ of them



So $x_0 < x_1 < \dots < x_{n+1}$.

Then there exist unique reals

$$c_0, c_1, \dots, c_n$$

s.t. $p(x) = c_0 + c_1 x + \dots + c_n x^n$

satisfies:

$p(x_0) = y_0$, $p(x_1) = y_1, \ldots, p(x_n) = y_n$.

Proof: We are trying to solve

$$c_0 + c_1 x_0 + c_2 x_0^2 + \ldots + c_n x_0^n = y_0$$

$$c_0 + c_1 x_1 + c_2 x_1^2 + \ldots + c_n x_1^n = y_1$$

$$\vdots$$

i.e.

$$A \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

Does this have a unique solution?

The following are equiv

(1) $A \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$ has a unique solution for all $y_0, \ldots, y_n$

(2) A is invertible

(3) $\det(A) \neq 0$

(4) A is of rank $n+1$

⋮

(36) The system (homogeneous version)

$$A\begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (*)$$

has $\begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ as its

unique solution.

=

Examine the last condition: if

$$p(x) = c_0 + c_1 x + \dots + c_n x^n$$

then $(*)$ holds iff

$$p(x_0) = 0, \ p(x_1) = 0, \ \dots, \ p(x_n) = 0$$

Claim: If so $p(x) =$ zero poly

Next time:

Prove claim w/o linear alg.