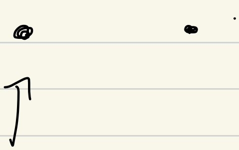
CPSC 303, Jan 19, 2024

- quasi-hybrid for today
- HW 2: Intro to MATLAB
- Celestial Mech
- $e^{A(t-t_0)}$

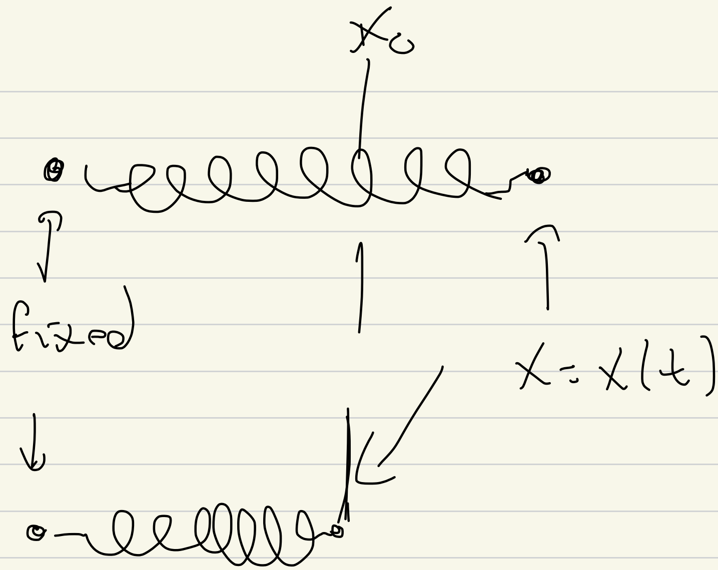
  when A is a matrix

Last time:

Harmonic oscillator:



idealized spring



at rest at x_0

and

$$\left\{\begin{matrix} x'' \\ \ddot{x} \\ \dfrac{d^2}{(dt)^2} x \end{matrix}\right\} = -C(x - x_0)$$

Simpler to take $x_0 = 0$

$x_0 = 0$?

$$x''(t) = -Cx, \quad C > 0$$

So $C = 1$

$$x''(t) = x$$

single variable
2nd order (x'')

Let $v(t) = x'(t)$

$$v' = -x$$

$$x' = v$$

$$\begin{bmatrix} v \\ x \end{bmatrix}' = \begin{bmatrix} v' \\ x' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix}$$

$$y' = Ay \qquad \vec{y}(t) = \begin{bmatrix} v(t) \\ x(t) \end{bmatrix}$$

$$\vec{y}\,' = \underbrace{A}_{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \vec{y}$$

Sol: 1-var: add $y(t_0) = y_0$

then $y(t) = e^{A(t-t_0)} y_0$

so --~

$$\vec{y}(t) = e^{A(t-t_0)} \vec{y}_0$$

Now! what is e^A

$$e^{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} = ??$$

Break ---

Celestial Mechanics:

$\vec{x}_1(t)$ $\vec{x}_2(t)$

$\vec{x}_3(t)$

$\vec{x}_n(t)$ ....

n-body problem ...

Say!

$\vec{x}_j(t)$

$\vec{x}_i(t)$

mass m_i

mass m_j

mass acc = Force

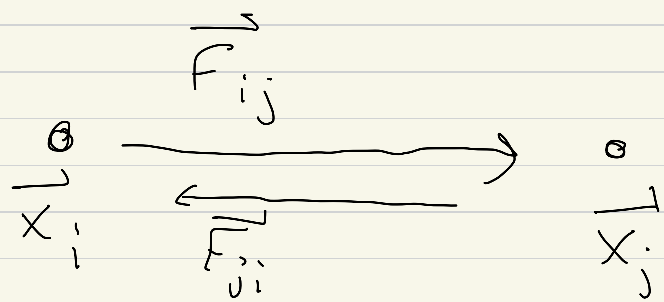
$$\boxed{m_i \underbrace{\vec{x}_i''(t)}_{\vec{a}_i(t)} = \sum_{j \neq i} \vec{F}_{ij}}$$

$$\vec{v}_i(t) = \vec{x}_i'(t)$$

where

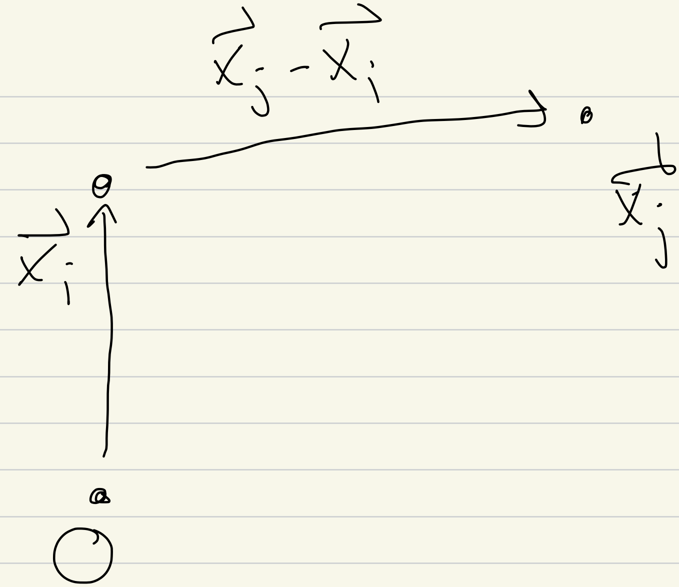
$$\vec{F}_{ij} = \vec{F}_{ij}(\vec{x}_1, \ldots, \vec{x}_n)$$

= force that body j exerts upon body i



Assumption: 1 $\vec{F}_{ij} = -\vec{F}_{ji}$

2 also proportional to $\vec{x}_i - \vec{x}_j$



=

Define:

$$\text{Momentum} = \sum_{i=1}^{n} m_i \vec{v}_i$$

$$= \sum_{i=1}^{n} m_i \vec{x}_i{}'$$

$$\text{Momentum}' = \sum_{i=1}^{n} m_i \vec{x}_i{}''$$

$$= \sum_{i=1}^{n} m_i \vec{a}_i$$

$$= \sum_{i=1}^{n} m_i \vec{a}_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j \neq i} \vec{F}_{ij} \right)$$

$$= \sum_{i \neq j} \vec{F}_{ij} =$$

$$\sum_{i < j} \left(\vec{F}_{ij} + \vec{F}_{ji} \right) = 0$$

e.g.

$$\vec{F}_{12} + \vec{F}_{13} + \vec{F}_{21} + \vec{F}_{23} + \vec{F}_{31} + \vec{F}_{32}$$

What is

$$e^{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

what is

$$e^{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

?

e^{A}, A is 2×2

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{A} = \mathbf{I} + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots$$

Method 1:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$A^3 = A^2 A = -IA = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^4 = I, \quad A^5 = A^4 A$$
$$= I \cdot A = A$$

$$A^6 = A^2, \ \ldots$$

More general way: similarity

We say A, B 2 $n \times n$ matrices are similar, if

$$A = S B S^{-1} \quad \text{for some } S$$

hopefully

$$A = S \begin{bmatrix} \text{very} \\ \text{simple} \\ \text{to work} \\ \text{with} \end{bmatrix} S^{-1}$$

$$A^2 = S B \underbrace{S^{-1} \; S}_{} B S^{-1}$$

$$= S B I B S^{-1}$$

$$= S B^2 S^{-1}$$

$$A^3 = SBS^{-1}A^2$$

$$= SBS^{-1}SB^2S^{-1}$$

$$= S\,B^3\,S^{-1}$$

Demo ...

S^{-1} = Joel moves from podium to blackboard

S = Joel moves from blackboard to podium

$$A_1 = S B_1 S^{-1}$$

$$A_2 = S B_2 S^{-1}$$

$$A_3 = S B_3 S^{-1}$$

Insted of $A_1 A_2 A_3$

$$= S B_1 S^{-1} S B_2 S^{-1} S B_3 S^{-1}$$

$$= S B_1 B_2 B_3 S^{-1}$$

S, T invert

$$(S T)^{-1} = T^{-1} S^{-1}$$

$$S T \, T^{-1} S^{-1}$$

$$S \, I \, S^{-1} = I$$

$$S T \, S^{-1} T^{-1} =$$