CPSC 303, Jan 17, 2024

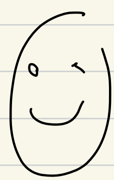
(Snow day)

- Introduce MATLAB

$|-3| \to$ error

abs(-3)

ans = 3



Today:

- Euler's Method
- MATLAB & Euler's method:
- $y' = Ay$, $y(t_0) = y_0$
- $y' = |y|^{1/2}$ Chaotic

$$y' = f(t, y)$$

(or $y' = f(y)$, or $\vec{y}\,' = \vec{f}(t, \vec{y})$,

...)

$y' = f(t, y)$ subject to

$$y(t_0) = y_0$$

Given $\underbrace{f = f(t, y)}_{\text{function}}$, t_0, y_0

$\uparrow \nearrow$

$\mathbb{R}$

$$y'(t) \approx \frac{f(t+h) - f(t)}{h}$$

h small

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$$

but ...

$$\frac{y(t+h) - y(t-h)}{2h}$$

much better approximation (typically)

$$y'(t) = f(t,y) \implies$$

given t, $y(t)$,

$$y'(t) = f(t,y) \approx \frac{y(t+h) - y(t)}{h}$$

$$y(t+h) \approx y(t) + h\,y'(t)$$

(Taylor's Thm)

$$= y(t) + h\,y'(t) + \frac{h^2}{2} y''(\xi)$$

ξ between t and $t+h$

$$y(t+h) = y(t) + h\,y'(t) + \frac{h^2}{2} y''(\xi)$$

$$= y(t) + h\,f(t,y) + \frac{h^2}{2} y''(\xi)$$

$$\approx \text{(h small)} \quad y(t) + h\,f(t,y)$$

Euler method

$$y(t+h) \approx y(t) + h\,f(t,y)$$

So $\quad t_0 =$ initial time

$\quad y_0 =$ initial value $y(t_0)$

Step 1: $\quad y(t_0) = y_0$

[Picking a value of h, smaller the better (usually)]

Step 2:

$$\underbrace{y(t_0+h)}_{y_1} = y_0 + h\,f(t_0, y(t_0))$$

Step 3:

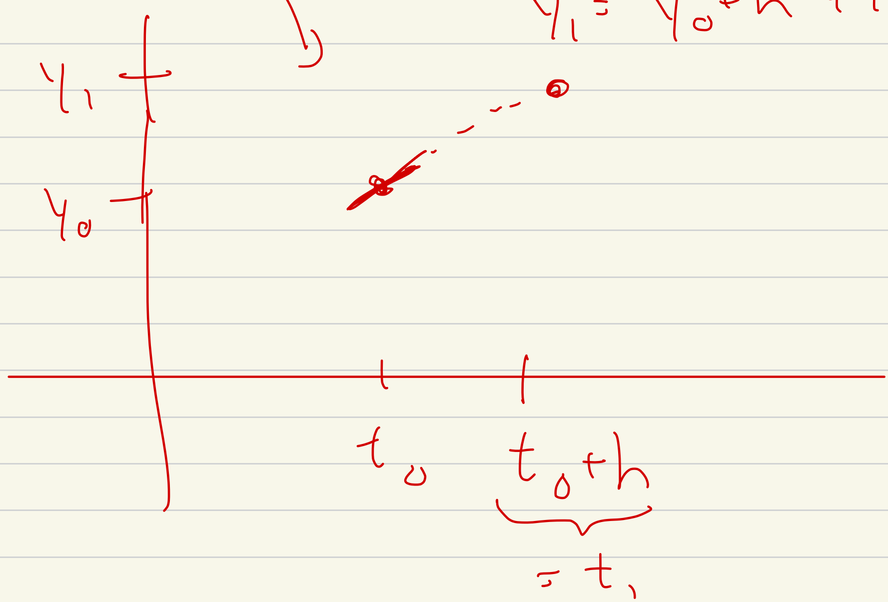
$$\underbrace{y(t_0+2h)}_{y_2} = y_1 + hf(t_1, y_1)$$

y_1 = given in Step 1

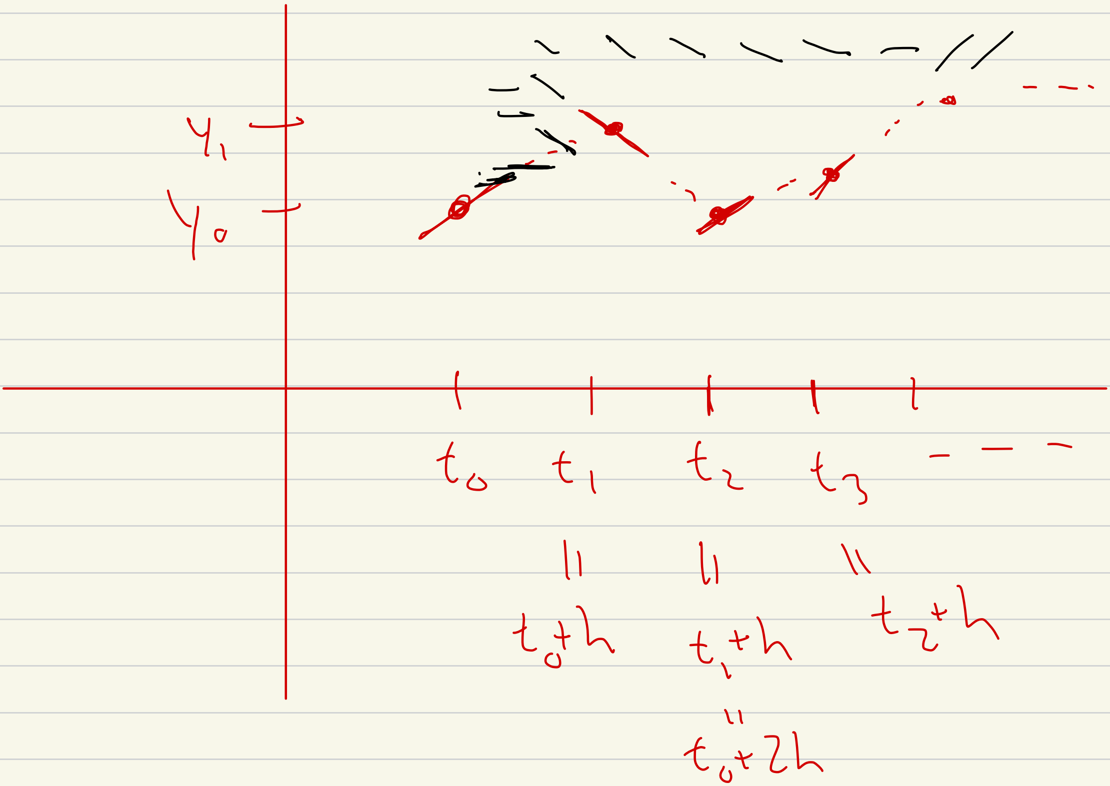
$t_1 = t_0 + h$

$$y' = f(t,y)$$

$$y_1 = y_0 + h\, f(t_0, y_0)$$



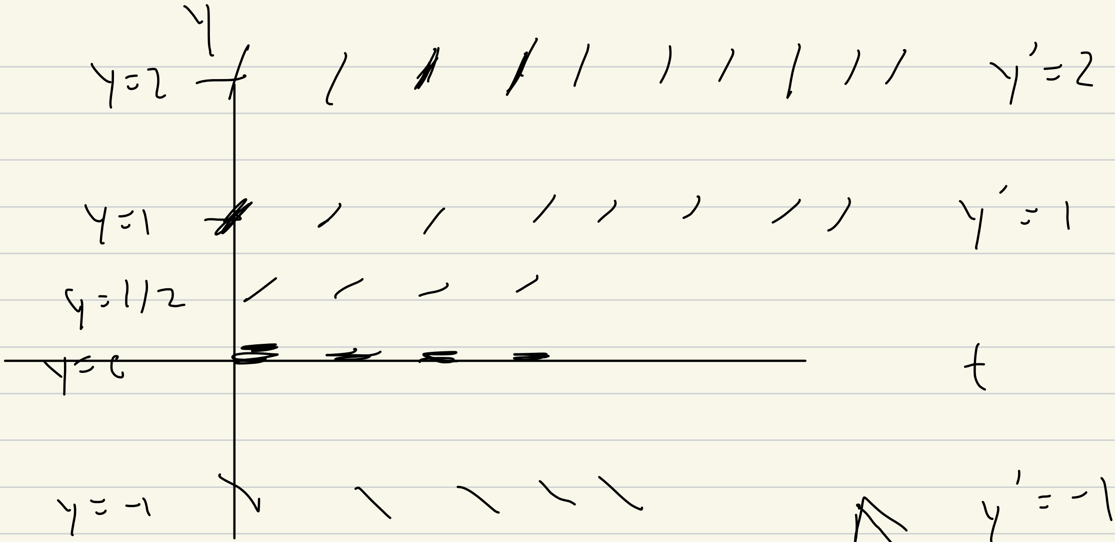
Isoclines



$$t_i = t_{i-1} + h$$

$$= i \cdot h + t_0$$

Actual ODE solvers try to adjust h, esp. if $f(t,y)$ is VERY large OR VERY quickly changing



indicate $y' = f(t,y)$

e.g. $y' = y$

Isocline picture

Given f, t_0, y_0

Pick small h

$t_1 = t_0 + h, \quad t_2 = t_0 + 2h, \ldots$

$y_1 \approx y_0 + h f(t_0, y_0)$

$y_2 \approx y_1 + h f(t_1, y_1)$

$\vdots$

$y' = Ay$ given y_0, t_0

Exact solution:

$$y(t) = e^{A(t-t_0)} y_0$$

$A = 2!$

$$y(t) = e^{2(t - t_0)} y_0$$

=

Num approx:

$$Y_1 = y_0 + h f(t_0, y_0)$$

$$= y_0 + h(A y_0)$$

OR

$A = 2$

$$= y_0 + h(2 y_0)$$

$$Y_1 = y_0(1 + 2h)$$

$$Y_2 = y_1 + h(2y_1) = (1+2h) y_1$$

$$= (1+2h)(1+2h)\, y_0$$

$$= (1+2h)^2\, y_0$$

Approx:

$y(t_i)$ or y_i

$$= (1+2h)^i\, y_0$$

$$y(t_0 + ih) = (1+2h)^i\, y_0$$

fix t_{end}:

t_{end}, N steps

$$t_0 + Nh = t_{end}$$

$$h = \frac{t_{end} - t_0}{N}$$

$=$

$$y(t_{end}) = (1+2h)^N y_0$$

$$= \left(1 + 2\frac{(t_{end} - t_0)}{N}\right)^N y_0$$

$$\boxed{= \left(1 + \frac{\text{something}}{N}\right)^N y_0}$$

$$\boxed{\overset{N\to\infty}{=} \quad e^{\text{something}} \, y_0}$$

$$= e^{z(t_{end} - t_0)} \, y_0$$

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$=$

See this in MATLAB,

$y' = |y|^{1/2}$ --