CPSC 303 Jan 15, 2024

Note:

1. Yennis' office hours today:

   2:30-4pm, ICCS X341

2. Preliminary version of notes

   on ODE's has been posted.

3. LaTeX snippets (warning...)

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Today: – Q2, Homework 1

– $\vec{y}\,' = A\vec{y} \qquad \vec{y}(t_0) = \vec{y}_0$

applied to $y'' = cy$ (harmonic oscillator)

– Euler's method

Say we have $(h>0)$

$f(x_0)$, $f(x_0 \pm h) = f(x_0+h)$,
$f(x_0-h)$

Use 3 values to approximate

$f'(x_0)$ ...

Simplest: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$

So probably

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

$$f'(x_0) \approx \frac{f(x_0) - f(x_0-h)}{h}$$

Systematically: $c_{-1}, c_0, c_1 \in \mathbb{R}$

$$\frac{c_{-1} f(x_0-h) + c_0 f(x_0) + c_1 f(x_0+h)}{h}$$

$$\approx f'(x_0)$$

$$=$$

e.g. $c_{-1}=0,\ c_1=1,\ c_0=-1$ :

$$\frac{0\ f(x_0-h) + (-1) f(x_0) + 1\ f(x_0+h)}{h}$$

$$\rightsquigarrow \frac{f(x_0+h) - f(x_0)}{h}$$

[A&G]:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} + O(h)$$

$$\frac{1}{h}\left(\frac{h^2}{2} f''(\xi)\right)$$

$$= \frac{h}{2} f''(\xi)$$

$$= O(h)$$

or $\quad O(h)\, M_2$

where $\quad M_2 \geq \quad |f''(\xi)|$

over all $\xi$ relevant

Linear algebra: Taylor exp.

$$c_1 f(x_0+h) = c_1\Big( f(x_0) + h f'(x_0)$$

$$+ \frac{h^2}{2} f''(x_0) + \begin{Bmatrix} O(h^3) \\ O(h^3) M_3 \\ |c_1| \underbrace{O(h^3) M_3}_{\text{positive}} \end{Bmatrix}$$

$$c_0 f(x_0) = c_0 f(x_0)$$

$$c_{-1} f(x_0-h) = c_{-1}\Big( f(x_0) + (-h) f'(x_0)$$

$$+ \frac{h^2}{2} f''(x_0) + O(h^3) M_3 |c_{-1}|\Big)$$

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Sum:

$$f(x_0)\left(c_1 + c_0 + c_{-1}\right) +$$

$$h f'(x_0)\left(c_1 - c_{-1}\right) +$$

$$\frac{h^2}{2} f''(x_0)\left(c_1 + c_{-1}\right) + O(h^3)$$

to have equal $h f'(x_0)$

$$+ 0 \, f(x_0) + 0 \, f''(x_0)$$

we want

$$c_1 + c_0 + c_{-1} = 0 \leftarrow f$$

$$c_1 - c_{-1} = 1 \leftarrow f'$$

$$c_1 + c_{-1} = 0 \leftarrow f''$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_0 \\ c_{-1} \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

solve ...

we solve (you solve)

$$c_1 = 1/2$$

$$c_0 = 0$$

$$c_{-1} = -1/2$$

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$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & & & \\ x_1^{n-1} & & \cdots & x_n^{n-1} \end{bmatrix}$$

"Vandermonde" matrix

interpolation, derivatives

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$\Longrightarrow$

$$\frac{\left(-\frac{1}{2}\right) f(x_0-h) + \left(\frac{1}{2}\right) f(x_0+h)}{h}$$

$$= \frac{h f'(x_0) + O(h^3)\, m_3 \left(|c_{-1}|+|c_0|+|c_1|\right)}{h}$$

$$= f'(x_0) + O(h^2) \begin{cases} \text{leave it} \\ m_3 \\ m_3\left(|c_{-1}| + |c_0| + |c_1|\right) \end{cases}$$

Solve! $\vec{y}' = A\vec{y}, \quad \vec{y}(t_0) = \vec{y}_0$

solution

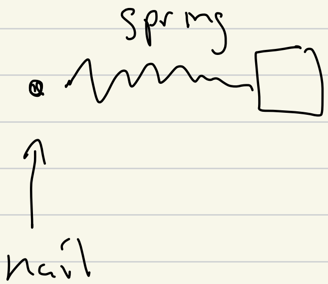
$$\vec{y}(t) = e^{A(t-t_0)}\,\vec{y}_0$$

as an

- system of $m$ ODE
- an ODE $m$-dimensional
- $A$ = $m \times m$ matrix of constants
- $\vec{y}(t) : \mathbb{R} \to \mathbb{R}^m$

time variables

# Harmonic Oscillator:




$x = x(t)$ position

$x : \mathbb{R} \to \mathbb{R}$

mass · acceleration = Force

acc. = Force/m

$$\left.\begin{array}{l} \dfrac{d^2}{dt^2}(x) = \\[2ex] x'' \\[2ex] \ddot{x} = \end{array}\right\} = -c\,(x - x_{rest})$$

$(c > 0)$

Simplify:

$$x'' = -Cx$$

$$x'' = -x$$

$$x''(t) = -x(t)$$

e.g.

$$(\sin(t))'' = -\sin(t)$$

$$(\cos(t))'' = -\cos(t)$$

guess: maybe general solution is

$$k_1 \sin(t) + k_2 \cos(t) \ \ldots \ ?$$

Idea: $v = x'$

$$v' = -x$$
$$x' = v$$

$$\begin{bmatrix} v \\ x \end{bmatrix}' = \begin{bmatrix} -x \\ v \end{bmatrix}$$

$$\vec{y} = (y_1(t), y_2(t)) : \underset{\text{time}}{\mathbb{R}} \to \mathbb{R}^2$$

$$y_1(t) = x(t)$$
$$y_2(t) = v(t)$$

$$\begin{bmatrix} v \\ x \end{bmatrix}' = \begin{pmatrix} -x \\ v \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} v \\ x \end{bmatrix}$$

$$\vec{y}\,' = A \vec{y}$$

Solution:

$$\vec{y}(t) = \underbrace{e^{A(t-t_0)}}_{\text{what is}} \vec{y}_0$$

$$e^{\text{matrix}}$$

to be continued ...