CPSC 303, Jan. 12, 2024

Today:

- ODE: $y' = y$, "isoclines"

- ODE's of the form

$$\boxed{y' = f(y)}$$

$y' = dy/dt$

$y = y(t)$

later $\vec{y}\,' = \vec{f}(t, \vec{y})$, system of m ODE's $\vec{y} = \vec{y}(t) : \mathbb{R} \to \mathbb{R}^m$

$\vec{f}(t, y) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$

- Examples: $y' = y^2$, $y' = |y|^{1/2}$

Question: What could possibly go wrong?

Executive summary:

Solve $y' = f(y) \quad y(t_0) = y_0$

① If f continuous near $y = y_0$, then a solution exists locally

② Moreover, if f is Lipschitz (or differentiable) near $y = y_0$, the local solution is unique

③ If f is analytic near $y = y_0$, the unique local solution is analytic

④ If $|f(y)| \leq Ky$ for some K constant, then there is a <u>global</u> solution.

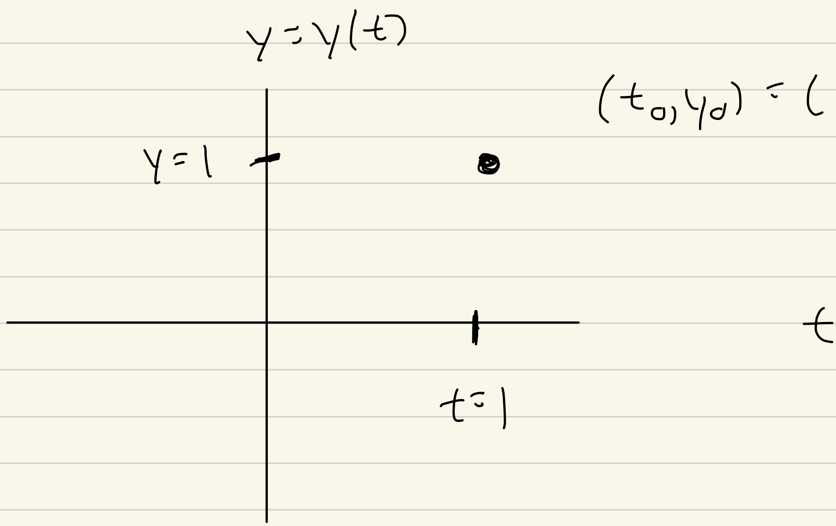
Last time:

initial condition

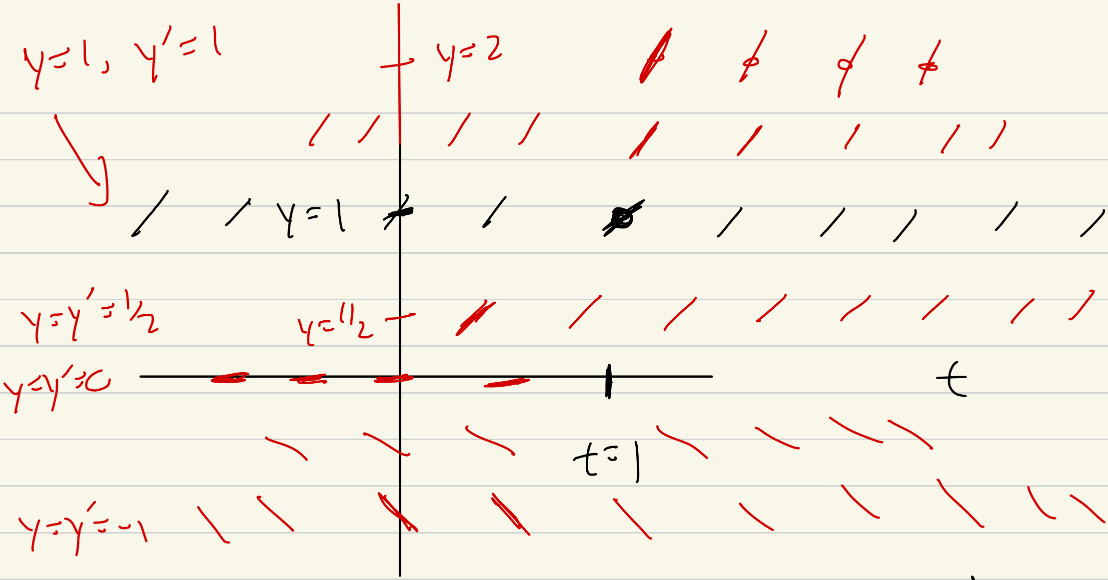
$$\boxed{y' = Ay, \quad y(t_0) = y_0}$$

$$y : \mathbb{R} \to \mathbb{R}$$

$(t_0, y_0) = (1, 1)$



Want: $\quad y' = y, \quad y(1) = 1$



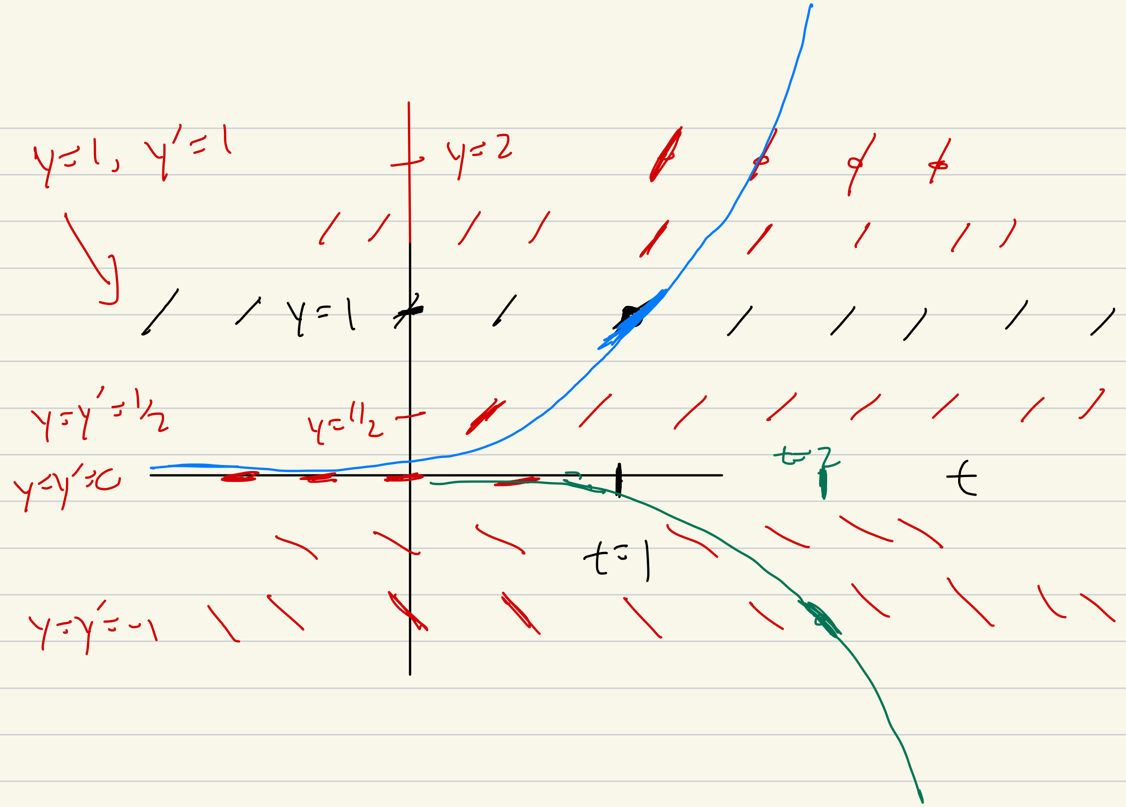
$y' = y$ $\qquad y = y(t),$ $\qquad y'(t) = \frac{d}{dt}(y)$

$= y(t)$

$y(t) = 1$	$y'(t) = 1$
$y(t) = \frac{3}{2}$	$y'(t) = 3/2$

$y' = y$: $\quad y(t) = e^{t+C}$

$y(t_0) = y_0$ $\qquad y(t) = e^{A(t-t_0)} y_0$



$y(t_0) = y_0$: $y(1) = 1$

$y(t) = e^{t-t_0} y_0 = e^{t-1} \cdot 1 = e^{t-1}$

$y(2) = -1$ $t_0 = 2,\ y_0 = -1$

$e^{t-t_0} y_0 = (-1) e^{t-2} = -e^{t-2}$

$$y' = y : \quad \frac{dy}{dt} = y$$

$$\frac{dy}{y} = dt$$

$$\int \frac{dy}{y} = \int dt$$

$y > 0$:
$$\int \frac{dy}{y} = \ln(y) + C_1$$

$$\int dt = t + C_2$$

$$\ln(y) + C_1 = t + C_2$$

$$\ln(y) = t + C$$

$$e^{\ln(y)} = e^{t+C}$$

$$y(t) = y = e^{t+C}$$

$y(t_0) = y_0 \Rightarrow$ determines C

$$\Rightarrow y(t) = e^{t-t_0} y$$

Consider $y' = y^2$, $y(1) = 1$

$$\frac{dy}{dt} = y^2$$

$$\frac{dy}{y^2} = dt$$

$$\int \frac{dy}{y^2} = \int dt$$

$$\frac{-1}{y} = t + C$$

$$y = \frac{-1}{t+C}$$

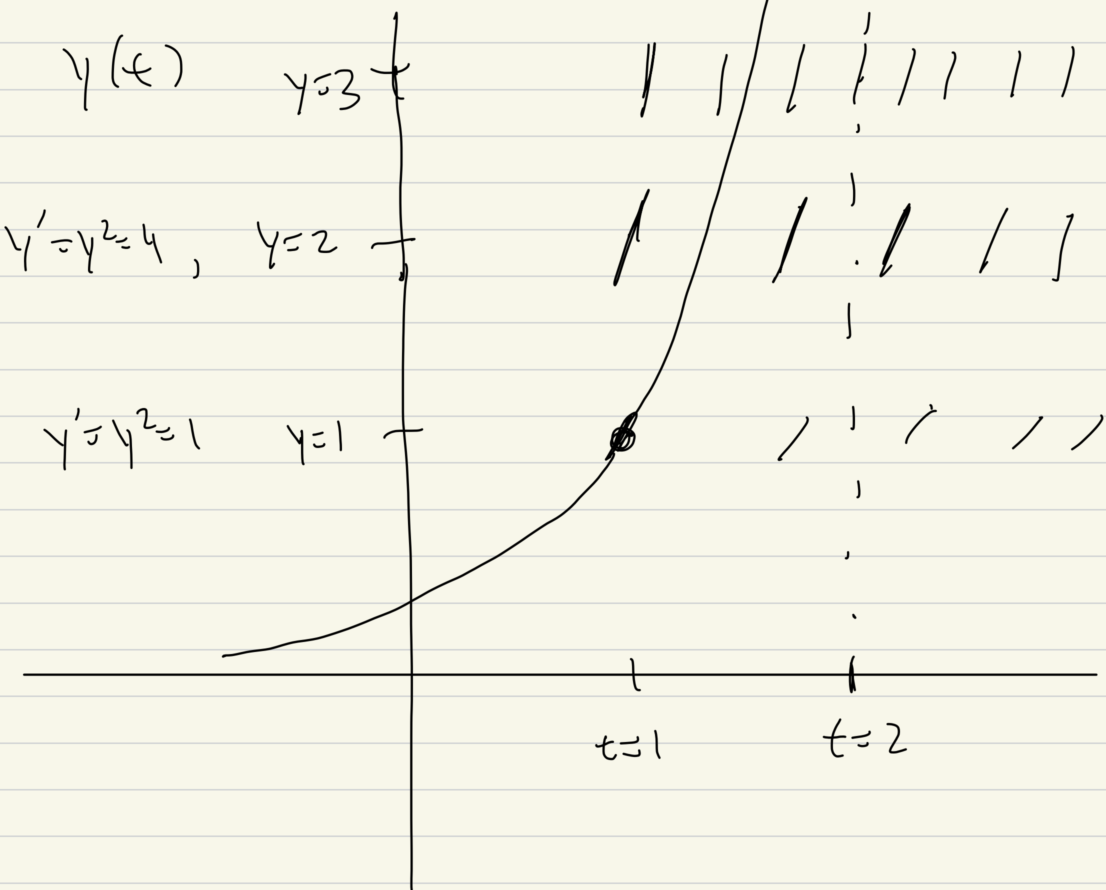
(Example 2.4.2 Calc 2)

$y(1) = 1 \qquad t_0 = 1, y_0 = 1$

$$1 = \frac{-1}{1+C} \qquad 1 + C = -1$$

$$C = -2$$

$$y = \frac{-1}{t-2} = \frac{1}{2-t}$$

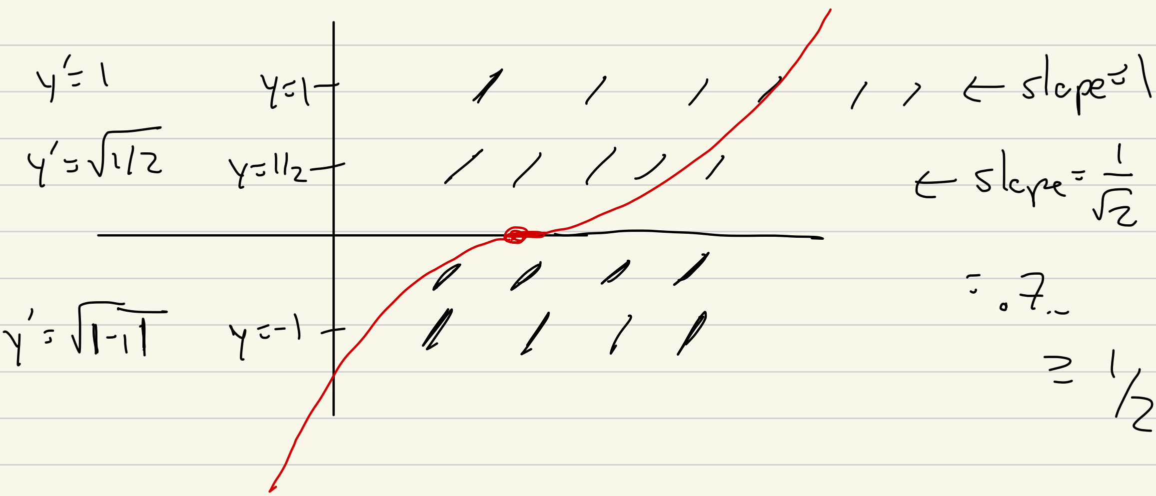


$$y(t) = \frac{1}{2-t}, \quad \boxed{\begin{array}{c} y(t) \to \infty \\ \text{as } t \to 2 \end{array}}$$

$$\boxed{y' = y^2}$$

What $y' = |y|^{1/2}$?

What could possibly go wrong?



$$\frac{dy}{dt} = |y|^{1/2}$$

$y > 0$: $$\frac{dy}{dt} = y^{1/2}$$

$$\int \frac{dy}{y^{1/2}} = \int dt \qquad \frac{1}{y^{1/2}} = y^{-1/2}$$

$$\frac{y^{1/2}}{1/2} = t + C$$

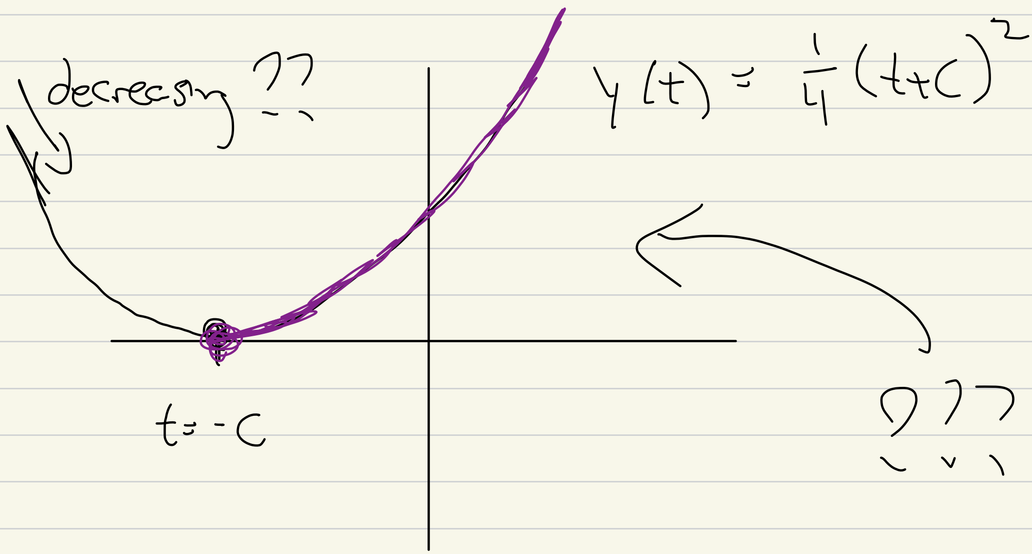
$$y^{1/2} = \frac{1}{2}(t+C)$$

$$y = \frac{1}{4}(t+C)^2$$

$$y' = \frac{1}{4}\left(2(t+C)\right) = \frac{1}{2}(t+C)$$

$$= \sqrt{\frac{1}{4}(t+C)^2} = \sqrt{y}$$

$$y(t) = \frac{1}{4}(t+C)^2$$



$$y' = |y|^{1/2} \geq 0$$

$y < 0$:
$$\frac{dy}{dt} = |y|^{1/2} = (-y)^{1/2}$$

$$\frac{dy}{(-y)^{1/2}} = dt$$

$(y<0)$

$$\int \frac{dy}{(-y)^{1/2}} = \int dt$$

$\|$

$$\int (-y)^{-1/2}\, dy$$

$$= \frac{(-y)^{1/2}}{1/2}(-1) = t + C$$

$$(-y)^{1/2}(-1) = \frac{1}{2}(t+C)$$

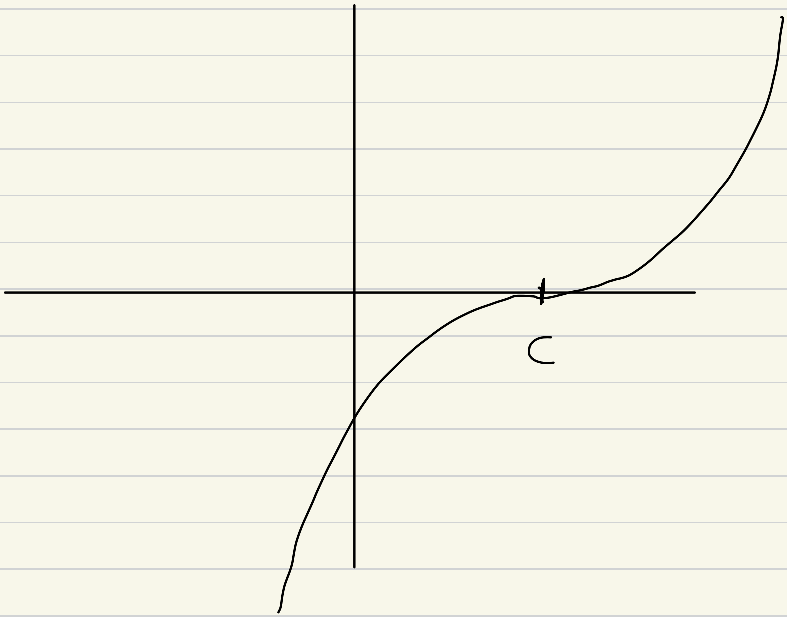
$$-y = \frac{1}{4}(t+C)^2$$

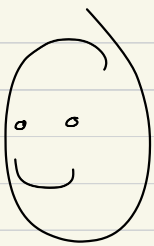
$$y = -\frac{1}{4}(t+C)^2$$

Our solutions:

$y > 0$: $\quad y(t) = \frac{1}{4}(t+C)^2$

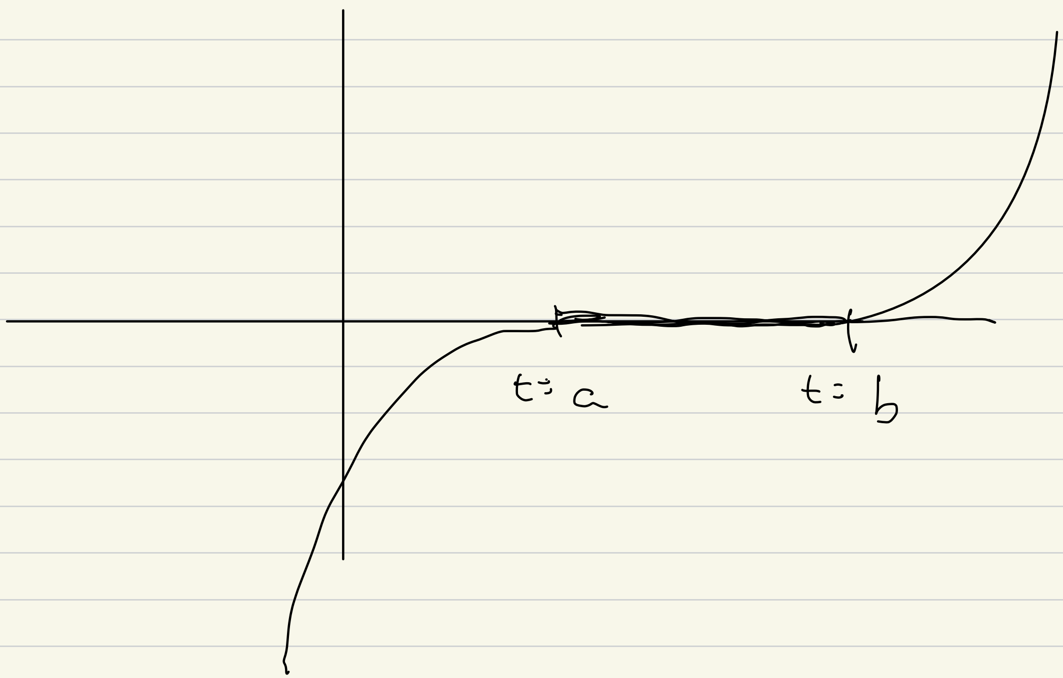
$y < 0$: $\quad y(t) = \frac{-1}{4}(t+C)^2$





$y = \frac{1}{4}(t+C)^2 \qquad y' = \frac{1}{4}\,2(t+C) = \frac{1}{2}(t+C)$

Claim: let $a < b$



Then:

$$y(t) = \begin{cases} \frac{1}{4}(t-b)^2 & b \le t \\ 0 & a \le t \le b \\ -\frac{1}{4}(t-a)^2 & t \le a \end{cases}$$

This is a solution of $y' = |y|^{1/2}$

Problem: $y' = f(y)$

$$f(y) = |y|^{1/2}$$