CPSC 303 Jan 10, 2024

- HW 1 will be assigned on Jan 11, due on gradescope on Jan 18

- Access gradescope via Canvas

- Today:

(1) Separable ODE's

(see, e.g., UBC Math Textbook

https://personal.math.ubc.ca/~CLP/CLP2

§ 2.4 Seperable Diff. Eq.

Ch 1: "Reviewing" terminology

Ch 4: $\|\vec{u}\|_2 = \sqrt{u_1^2 + \dots + u_n^2}$

for $\vec{u} \in \mathbb{R}^n$

- Classes of functions
- Taylor Series
- ODE's:

ODE's basic foundation same big field $y' = f(t, y)$

PDE $h = h(t, x_1, \dots, x_n)$

heat eq $\frac{\partial h}{\partial t} = -\Delta_{x_1, \dots, x_n} h$

PDE's   foundations of elliptic PDEs

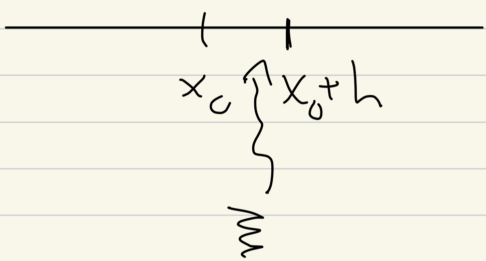
"   " parabolic "

Taylor's Thm: $|h|$ small intuitively

$$f(x_0+h)$$

$$= f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(\xi)$$

$$\approx f(x_0) + h\, f'(\text{somewhere in between } x_0,\ x_0+h)$$



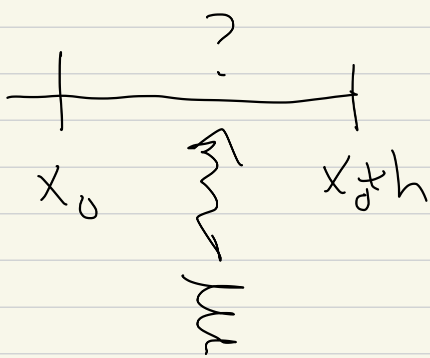
Mean Value Thm

$$f(x_0+h) =$$

$$f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0)$$

$$+ \frac{h^3}{3!} f^{(3)}(x_0) + \cdots$$

$$+ \frac{h^k}{k!} f^{(k)}(x_0) + \frac{h^{k+1}}{(k+1)!} f^{(k+1)}(\xi)$$



$$f(x_0+h) = f(x_0) + h f'(x_0)$$

$$+ \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f'''(\xi_1)$$

$$h f'(x_0) = f(x_0+h) - f(x_0)$$

$$- \frac{h^2}{2} f''(\xi_2)$$

$$\boxed{\begin{aligned} f'(x_0) &= \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(\xi_2) \\ &= \frac{f(x_0+h) - f(x_0)}{h} + \mathrm{Order}(h) \end{aligned}}$$

where

$O(h)$ is some function

(depends $f, x_0,$)

$$\boxed{\left|\text{Order}(h)\right| \le \frac{h}{2} M_2}$$

where M_2 bound on f''

in the interval $[x_0, x_0+h]$

By def:

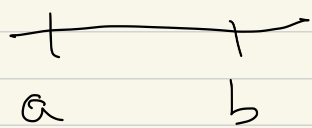
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Really taken from

[A&G], Ch 14, Sections 1,2

Useful notation:

If $a < b \in \mathbb{R}$



$$C[a,b] \overset{\text{def}}{=} \{ f : [a,b] \to \mathbb{R} \text{ s.t. } f \text{ is continuous} \}$$

$$k = \mathbb{N} = \{1, 2, 3, \dots\}$$

$$C^k(a,b) = \{ f: (a,b) \to \mathbb{R}$$

s.t. f has k continuous

derivatives for all $x \in (a,b) \}$

$$C^k[a,b] = \{ f: [a,b] \to \mathbb{R} \text{ s.t.}$$

s.t. f has k continuous

derivatives for all $x \in [a,b] \}$

$$(a,b) = \{ x \in \mathbb{R} \mid a < x < b \}$$

$$[a,b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

such that

$$C^{\infty}(a,b) = \{ f : (a,b) \to \mathbb{R} \text{ s.t. }$$

$$f \text{ has all derivatives of all order} \}$$

$$C^{\omega}(a,b) = \{ f : (a,b) \to \mathbb{R} \text{ s.t. }$$

for all $x_0 \in (a,b)$,

$$f(x_0+h)$$

$$= f(x_0) + \frac{h}{1} f'(x_0) + \frac{h^2}{2!} f''(x_0)$$

$$+ \frac{h^3}{3!} f'''(x_0) + \cdots$$

For example

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cdots + \frac{x^n}{n!} + \cdots$$

"Taylor series"

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

[This is the case $x_0 = 0$, $h \mapsto x$]

Based on $(e^x)' = e^x$, $(\sin x)' = \cos x$,

$\cdots$

$C^0(a,b)$

$$C^1(a,b) \subset C^0(a,b)$$

$$C^2(a,b) \subset C^1(a,b) \subset C^0(a,b)$$
$$= C(a,b)$$

C^ω (real analytic)

$$\subset C^\infty \subset \dots \subset C^4 \subset C^3 \subset C^2 \subset C^1$$
$$\subset C^0$$

Start ODE!

Simple ODE [A&G]:

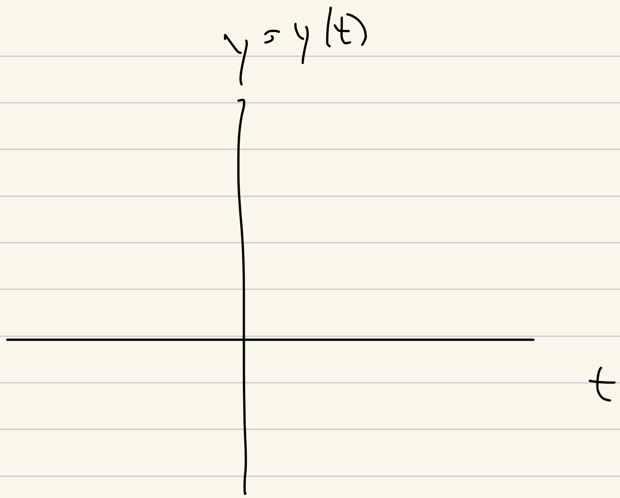
$$y' = f(t, y)$$

$$y' = \frac{dy}{dt} \equiv \dot{y}$$

Caution! Math textbooks:

$$y' = \frac{dy}{dx} = f(x, y)$$

$y = y(t)$



$$\frac{dy}{dt}(t) = y'(t)$$

$$= \text{given function}\,(t, y(t))$$

Given "initial condition"

$t_0, y_0 \in \mathbb{R}$ impose

$$y(t_0) = y_0$$

we expect a unique solution.

$$\boxed{y'(t) = A\, y(t)}$$

$A \in \mathbb{R}$, A given

Claim:

$$y(t) = e^{At}\, C$$

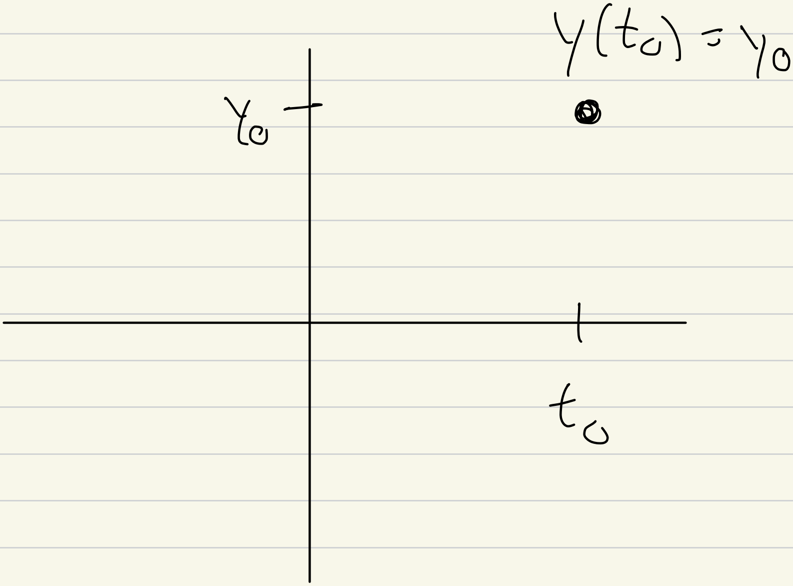
verify:

$$y'(t) = \left(e^{At}\, C\right)'$$

$$= \left(A\, e^{At}\right) C$$

$$= A\, y(t)$$

$y(t_0) = y_0$ given $y_0, t_0 \in \mathbb{R}$



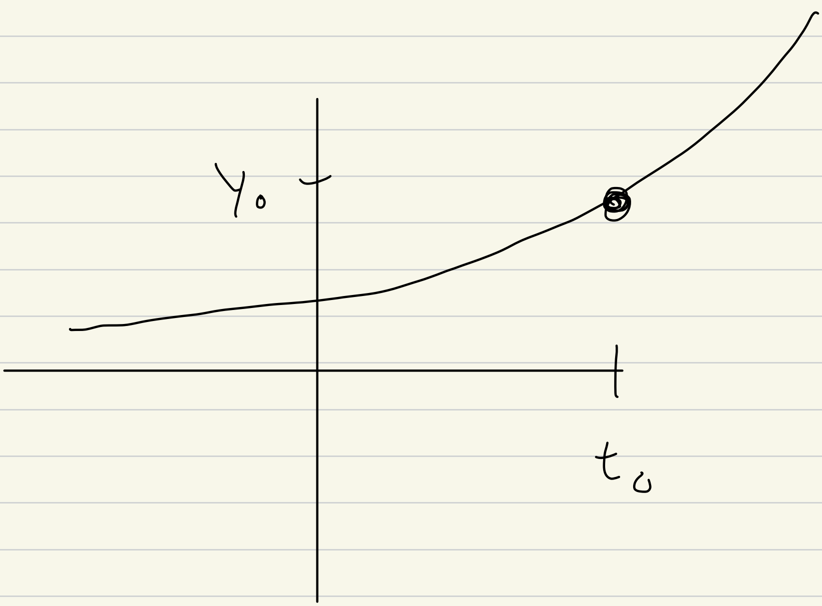
$$y(t_0) = e^{At_0} C$$

$$\begin{array}{c} \shortparallel \\ y_0 \end{array} \qquad \boxed{C = y_0 e^{-At_0}}$$

$$y(t) = e^{At} C$$

$$= e^{At} y_0 e^{-At_0}$$

$$= y_0 e^{A(t-t_0)}$$



A big enough, > 0

Is the solution unique ???

Can we simply guess ???

Class Ended:

$$f(x_0+h) = f(x_0) + h f'(x_0) + \dots$$

$$x_0 = 0$$

$$f(h) = f(0) + h f'(0) + \frac{h^2}{2} f''(0) + \dots$$

$$e^h = 1 + h \cdot 1 + \frac{h^2}{2} \cdot 1 + \dots$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots \quad \left(\begin{array}{c}\text{exchanged} \\ x \text{ for } h\end{array}\right)$$