CPSC Homework 9 solutions

Individual:

(1) For $y' = f(y) = 4y + 5$ with $y(6) = 7$

(a) Euler's method:

$$
\begin{aligned}
y_{n+1} &= y_n + h f(y_n) \\
&= y_n + h(4y_n + 5) \\
&= (1+4h) y_n + 5h
\end{aligned}
$$

(b) Explicit Trapezoidal:

$$\tilde{Y}_{n+1} = y_n + h f(y_n) = (1+4h) y_n + 5h$$

$$
\begin{aligned}
f(\tilde{Y}_{n+1}) &= f((1+4h) y_n + 5h) \\
&= 4\big((1+4h) y_n + 5h \big) + 5 \\
&= 4(1+4h) y_n + (5 + 20h) \\
&= (4 + 16h) y_n + (5 + 20h)
\end{aligned}
$$

$$y_{n+1} = y_n + h\,\frac{f(y_n) + f(\tilde{y}_n)}{2}$$

$$= y_n + h\,\frac{(4y_n + 5) + \left[(4+16h)y_n + 5 + 20h\right]}{2}$$

$$= y_n + \left(h\,\frac{2(4y_n+5) + h(16y_n + 20)}{2}\right)$$

$$= y_n(1 + 4h + 8h^2) + 5h + 10h^2$$

(c) Exact solution: Homogeneous: $y' = 4y$

so $y(t) = Ce^{4t}$. Guess specific: $y(t) = c_0$

(constant), so $y' - 4y = -4c_0$ which $= 5$

so $c_0 = -5/4$. So general solution:

$$y(t) = Ce^{4t} - 5/4. \quad y(0) = 1$$

$$\Rightarrow Ce^0 - 5/4 = 1 \Rightarrow C = 9/4$$

Hence $y' = 4y + 5, \quad y(0) = 1$

$$\Rightarrow \quad y(t) = \tfrac{9}{4} e^{4t} - \tfrac{5}{4} .$$

$$y(h) = \tfrac{9}{4} e^{4h} - \tfrac{5}{4}$$

$$= \tfrac{9}{4} \left(1 + 4h + \frac{(4h)^2}{2} + \frac{(4h)^3}{3!} + O(h^4) \right)$$

$$= 1 + 9h + 18h^2 + 24h^3 + O(h^4)$$

(d) $y_{n+1} = (1 + 4h) y_n + 5h$

So if $y(0) = 1$, then Euler's method

gives

$$y_1 = (1+4h)1 + 5h = 1 + 9h$$

as an approximation to

$y(h) = 1 + 9h + 72h^2$, which matches

up to $O(h^2)$.

(e) $y_{n+1} = y_n(1 + 4h + 8h^2) + 5h + 10h^2$

So $y_0 = 1$ gives

$$y_1 = (1 + 4h + 8h^2) + 5h + 10h^2$$

$= 1 + 9h + 18h^2$, which matches

$y(h)$ up to $O(h^3)$.

(f) Solve $y_{n+1} = a y_n + b$ where $a \neq 1, 0$

a, b constants:

Homogeneous: $y_{n+1} - a y_n = 0$, so $y_n = C a^n$

Particular: guess $y_n = c_0$ (constant). So

$c_0 = a c_0 + b$ so $c_0(1-a) = b$ so $c_0 = \frac{b}{1-a}$

So general solution:

$$y_n = C a^n + \frac{b}{1-a}$$

(C constant)

(g) Solve $y_{n+1} = (1+4h)y_n + 5h$:

In (c) : $a = 1+4h$

$b = 5h$

So $$y_n = C(1+4h)^n + \frac{5h}{1-(1+4h)}$$

$$= C(1+4h)^n - 5/4 .$$

$y_0 = 1 \Rightarrow 1 = C - 5/4 \Rightarrow C = 9/4$

So $$y_n = 9/4\,(1+4h)^n - 5/4$$

(h) Euler's method, with $h = \frac{1}{m}$ approximates

$y(2)$ as

$$y_{2m} = 9/4\,(1+4/m)^m - 5/4$$

Using $(1+\varepsilon) = e^{\varepsilon - \varepsilon^2/2 + O(\varepsilon^3)}$

we get this approximation to be

$$\tfrac{9}{4}\, e^{m\left(4/m - (4/m)^2/2 + O(1/m^3)\right)} - 5/4$$

$$= \tfrac{9}{4}\, e^{4 - 8/m + O(1/m^2)} - 5/4$$

$$= \tfrac{9}{4}\, e^4\, e^{-8/m + O(1/m^2)} - 5/4$$

$$= \tfrac{9}{4}\, e^4 \left(1 - 8/m + O(1/m^2)\right) - 5/4$$

$$= \underbrace{9/4\, e^4 - 5/4}_{\text{true solution}} \underbrace{- \tfrac{9}{4} e^4 \cdot 8/m + O(1/m^2)}_{\text{difference between Euler and true solution}}$$

$$= 18\, e^4/m + O(1/m^2)$$

$$= O(1/m)$$

(i) $y_{n+1} = y_n \underbrace{(1 + 4h + 8h^2)}_{a} + \underbrace{5h + 10h^2}_{b}$

gives

$$y_n = Ca^n + \frac{b}{1-a}$$

$$= C(1+4h+8h^2)^n + \frac{5h+10h^2}{-4h-8h^2}$$

$$= C(1+4h+8h^2)^n - \frac{5}{4}$$

So $y_0 = 1$ implies $1 = C - 5/4$

so $C = 9/4$ and

$$y_n = 9/4(1+4h+8h^2)^n - 5/4$$

Hence with $h = 1/m$

$$y_{2m} = 9/4\left(1 + 4/m + 8/m^2\right)^m - 5/4$$

and $$\left(1+\underbrace{4/m+8/m^2}_{\delta}\right) = e^{\delta-\delta^2/2+\delta^3/3+O(\delta^4)}$$

$$e^{\left[(4/m+8/m^2)-(4/m+8/m^2)^2/2+(4/m+8/m^2)^3/3+O\left(\frac{1}{m^4}\right)\right]}$$

$$= e^{4/m+8/m^2-\left(16/m^2+64/m^3\right)/2+4^3/3m^3+O\left(\frac{1}{m^4}\right)}$$

$$= e^{4/m-32/m^3+(64/3)/m^3+O(1/m^4)}$$

$$= e^{4/m-32/3m^3+O(1/m^4)}$$

So $$y_{2m} = \frac{9}{4}\underbrace{e^{4-32/3m^2+O(1/m^3)}}_{e^4\left(1-O\left(\frac{1}{m^2}\right)\right)} - \frac{5}{4}$$

Group Homework

(1) Joel Friedman

(2) If $S_0'''(x_1) = S_1'''(x_1)$ then

$$d_0 = d_1 .$$

We have $d_0 = \frac{c_1 - c_0}{3h_1}$, $d_1 = \frac{c_2 - c_1}{3h_2}$,

and for a natural spline we have $c_0 = c_2 = 0$.

So $d_0 = \frac{c_1}{3h_1}$, $d_1 = -c_1/3h_2$, so $d_0 = d_1 \Leftrightarrow c_1 = 0$

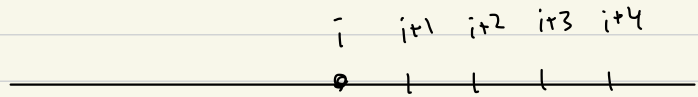
Since

$$1 \cdot c_1 = \frac{3}{2} f[x_0, x_1, x_2]$$

we have $v'''(x)$ is continuous thru

$x = x_1$ iff $f[x_0, x_1, x_2] = 0$

(3)



Walks from i:

$$i \to \begin{cases} i+1 & \begin{cases} i+2 \\ i \end{cases} \\ i-1 & \begin{cases} i \\ i-2 \end{cases} \end{cases} \quad \text{so} \quad \begin{array}{ll} \to i+2 & \text{one} \\ \to i & \text{two} \\ \to i-2 & \text{one} \end{array}$$

$i \pm 1$: 2 of length 1

$i+2,\ i,\ i,\ i-2$: 4 of length 2

Continuing: from length 2 walks to $i+2$

$$i+2 \begin{cases} i+3 & \begin{cases} i+4 \\ i+2 \end{cases} \\ i+1 & \begin{cases} i+2 \\ i \end{cases} \end{cases} \quad \text{so} \quad \begin{array}{ll} \to i+4 & \text{one} \\ \to i+2 & \text{two} \\ \to i & \text{one} \end{array}$$

length 2   length 3   length 4

Similarly, from 2 of length 2 walks to i $\Big\}$ $\begin{array}{ll} \to i+2 & \text{one} \cdot 2 \\ \to i & \text{two} \cdot 2 \\ \to i-2 & \text{one} \cdot 2 \end{array}$

Similarly from one of lengh 2 to $i-2$ $\Big\}$ $\begin{array}{ll} i & \text{one} \\ i-2 & \text{two} \\ i-4 & \text{one} \end{array}$

Total!

$i \to i+4$ one

$i \to i+2$ four

$i \to i$ six

$i \to i-2$ four

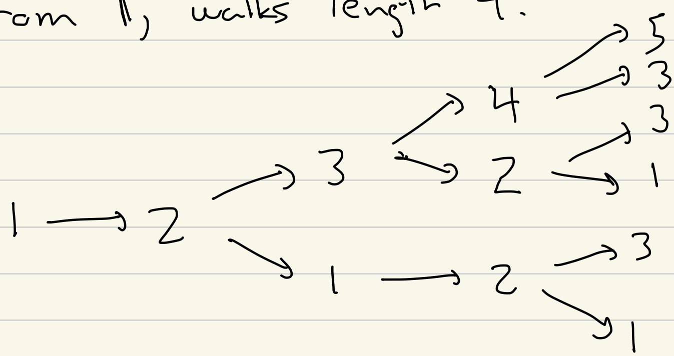
$i \to i-4$ one

So i^{th} row is

$$\cdots\ 1\ 0\ 4\ 0\ 6\ 0\ 4\ 0\ 1\ \cdots$$

$i-4^{th}$ col. $\quad$ $i-2^{nd}$ col $\quad$ i^{th} col $\quad$ $i+2^{nd}$ col $\quad$ $i+4^{th}$ col

From 1, walks length 4:



So 1^{st} row looks like

$$\left[\begin{matrix} 2 & 0 & 3 & 0 & 1 & 0 & 0 & \cdots \end{matrix}\right]$$