CPSC 2024 Homework 8 solutions

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2. (a) $p[x_2, x_1] = p[x_1, x_2]$

$$= \frac{(c_0 + c_1 x_2 + c_2 x_2^2) - (c_0 + c_1 x_1 + c_2 x_1^2)}{x_2 - x_1}$$

$$= \frac{c_1(x_2 - x_1) + c_2(x_2^2 - x_1^2)}{x_2 - x_1}$$

$$= c_1 + c_2(x_2 + x_1)$$

Similarly

$$p[x_1, x_0] = c_1 + c_2(x_1 + x_0)$$

$$p[x_2, x_1] - p[x_1, x_0] = c_2(x_2 + x_1 - x_1 - x_0)$$

$$= c_2(x_2 - x_0)$$

2(b) Hence

$$c_2 = \frac{p[x_2, x_1] - p[x_1, x_0]}{x_2 - x_0}$$

3(a)

$$\text{Energy}_{2,w}(v + \varepsilon g) =$$

$$\int_A^B w(x) \left(v'' + \varepsilon g'' \right)^2 dx$$

$$= \int_A^B w(x) \left((v'')^2 + 2\varepsilon (v'')(g'') + \varepsilon^2 (g'')^2 \right) dx$$

$$= \text{Energ}_{2,w}(v) + \varepsilon^2 \, \text{Energy}_{2,w}(g)$$

$$+ 2\varepsilon \int_A^B w(x) \, v''(x) \, g''(x) \, dx$$

(b) If $\alpha \leq \alpha + \varepsilon \beta + \varepsilon^2 \gamma$

then $f(\varepsilon) = \alpha + \varepsilon\beta + \varepsilon^2\gamma$ has a local minimum at $\varepsilon = 0$, so

$$f'(0) = (\beta + 2\varepsilon\gamma)\Big|_{\varepsilon=0} = \beta$$

must be 0. Hence

$$\int_A^B w(x)\, v''(x)\, g''(x)\, dx = 0$$

3(c) Hence

$$0 = \int_A^B w(x)\, v''(x)\, g''(x)\, dx =$$

$$= \big(w(x)v''(x)\big)\, g'(x)\Big|_A^B - \int_A^B \big(w(x)v''(x)\big)'\, g'(x)\, dx$$

$$= 0 - \int_A^B \big(w(x)v''(x)\big)'\, g'(x)\, dx$$

since $g'(A) = g'(B) = 0$. Integrating by parts again:

$$0 = \int_A^B \left(w(x)v''(x)\right)' g'(x)\,dx$$

$$= w(x)v''(x)g(x)\Big|_A^B - \int_A^B \left(w(x)v''(x)\right)'' g(x)\,dx$$

$$= 0 - \int_A^B \left(w(x)v''(x)\right)'' g(x)\,dx$$

Since $g(A) = g(B) = 0$. Hence

$$\int_A^B \left(w(x)v''(x)\right)'' g(x)\,dx = 0$$

3(d) By the argument given in class

if $\left(w(x)v''(x)\right)'' \neq 0$ for some $x_0 < x < x_1$

[then $w(x)V''(x) > 0$ (or < 0) for all x near some $\xi \in (x_0, x_1)$. Taking $g(x)$ to be > 0 near ξ and otherwise 0 we have $\int_A^B (w(x)V''(x))'' g(x)\,dx > 0$ (or < 0), a contradiction. Hence]

$$\left(w(x)V''(x)\right)'' = 0 \quad \text{for all } x_0 < x < x_1$$

[Note! the material in brackets above, (i.e., [,]) is optional to write, as was explained in class.]

3(e) If $w(x) = 1$ everywhere, then

$(w(x)V''(x))'' = 0$ for all $x_0 < x < x_1$

implies $V''''(x) = 0$ there, hence

$V'''(x)$ is a cubic polynomial.

3(f) More generally: $(w(x)V''(x))'' = 0$

implies that $(w(x)V''(x))' = c_1$, and

hence $w(x)V''(x) = c_1 x + c_0$, for

constants c_1, c_0. Hence

$$V''(x) = \frac{c_0 + c_1 x}{w(x)}$$

provided that $w(x)$ is nowhere 0.

4 (a) For $n \geq 3$ $\quad N_{rod,n} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ & & & & \\ & & & & \end{bmatrix} \leftarrow$ this row has sum of absolute values $= 2$

and $N_{rod,n}$ has each row having at most 2 1's and the rest 0's.

Hence for $n \geq 3$, $\|N_{rod,n}\|_\infty = 2$.

For $n=2$, $\|N_{rod,2}\|_\infty = \left\| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\|_\infty = 1$

and $N_{rod,1} = ?\ [0]\ ? \ldots$ (probably $N_{rod,1}$ is best viewed as $[0]$, but this is not so important...)

(b) $(4 I_n + N_{rod,n})^{-1} = \left(4\left(I_n + \frac{1}{4} N_{rod,n}\right)\right)^{-1}$

$= \frac{1}{4}\left(I_n + \frac{1}{4} N_{rod,n}\right)^{-1} = \frac{1}{4}\left(I_n - \left(\frac{-1}{4} N_{rod,n}\right)\right)^{-1}$

$= \frac{1}{4}\left(I_n - \frac{1}{4} N_{rod,n} + \frac{1}{16} N_{rod,n}^2 - \cdots\right)$

or $\frac{1}{4} \sum_{k=0}^{\infty} \left(\frac{-1}{4} N_{rod,n} \right)^k$

(c) Some experimentation suggests

$$(N_{rod,n})^2 = \begin{bmatrix} 1 & 0 & 1 & & & \text{O's} \\ 0 & 2 & 0 & 1 & & \\ 1 & 0 & 2 & \ddots & 1 & \\ & 1 & \ddots & \ddots & 0 & 1 \\ \text{O's} & & \ddots & & 2 & 0 \\ & & & 1 & 0 & 1 \end{bmatrix}$$

i.e. for large n:

$$(N_{rod,n})^2_{ij} = \begin{cases} 1 & \text{if } |i-j| = 2 \\ 2 & \text{if } i=j \text{ and } i \neq 1,n \\ 1 & \text{if } i=j=1 \text{ or } i=j=n \\ 0 & \text{otherwise} \end{cases}$$

(d) Experiments suggest:

$$(N_{rod,n})^3 = \begin{bmatrix} 0 & 2 & 0 & 1 & & & & & \text{O's} \\ 2 & 0 & 3 & 0 & 1 & & & & \\ 0 & 3 & 0 & 3 & 0 & \ddots & & & \\ 1 & 0 & 3 & 0 & \ddots & & 1 & & \\ & 1 & 0 & \ddots & & & 0 & 1 & \\ & & 1 & \ddots & & & 3 & 0 & 1 \\ & & & \ddots & & & 0 & 3 & 0 \\ \text{O's} & & & & 1 & 0 & 3 & 0 & 2 \\ & & & & & 1 & 0 & 2 & 0 \end{bmatrix}$$

i.e. for large n,

$$(N_{rod,n})^3_{ij} = \begin{cases} 1 & \text{if } |i-j| = 3 \\ 3 & \text{if } |i-j| = 1 \text{ and } i+j \neq 3, 2n-1 \\ 2 & \text{if } (i,j) = (1,2),(2,1),(n,n-1),(n-1,n) \\ 0 & \text{otherwise} \end{cases}$$