Homework Solutions 7

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(2) (a) Ans: absolute error $7.99\ldots \times 10^{-15}$

relative error $5.08\ldots \times 10^{-15}$

[If $\text{trueVal} = \dfrac{\sqrt{2}+\sqrt{3}}{2}$ and

$\text{monoVal} = c_0 + (2.005)c_1$ where MATLAB

finds $\vec{c} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$ via

$$\vec{c} = A^{-1}\begin{bmatrix} \sqrt{2} \\ \sqrt{3} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 2.01 \end{bmatrix},$$

then

$$\text{absolute error} = |\text{trueVal} - \text{monoVal}|$$

and

$$\text{relative error} = \frac{|\text{trueVal} - \text{monoVal}|}{|\text{trueVal}|}.$$
]

2(b) absolute error $= 6.16\ldots \times 10^{-11}$

relative error $= 3.91\ldots \times 10^{-11}$

2(c) absolute error $= 1.28\ldots \times 10^{-6}$

relative error $= 8.15\ldots \times 10^{-7}$

2(d) cond(A, Inf) in MATLAB gives

$1.19\ldots \times 10^{11}$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2.0\ldots01 \end{bmatrix}, \text{ so}$$

$$\left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\|_\infty = \max(|a|+|b|, |c|+|d|) \approx 3$$

$$A^{-1} = 10^{10} \begin{bmatrix} 1.99\ldots & -1.99\ldots \\ -0.99\ldots & 0.99\ldots \end{bmatrix}$$

So $\|A^{-1}\|_\infty \approx 10^{10} \cdot 4$

so $\|A\|_\infty \|A^{-1}\|_\infty \approx 10^{10} \cdot 12 = 1.2 \times 10^{11}$

2(e) $1.2 \times 10^{11} \cdot 1.11\ldots \times 10^{-16} \approx 1.3 \times 10^{-7}$.

2(f) absolute error $= 0$

relative error $= 0$

2(g) absolute error $= 2.22\ldots \times 10^{-16}$

relative error $= 1.46\ldots \times 10^{-16}$

3(a) (i) $\alpha_2 = 1.5498596\ldots$

(ii) $\text{cond}(A, \text{Inf}) = 5.37\ldots \times 10^5$

(b) (i) $\alpha_6 = 1.5492553\ldots$

(ii) $\text{cond}(A, \text{Inf}) = 5.59\ldots \times 10^{13}$

(c) (i) $\alpha_7 = 1.5156250000000000$

(ii) $\text{cond}(A, \text{Inf}) = 5.48\ldots \times 10^{15}$

(d) (i) $\alpha_8 = \text{Inf}$

(ii) $\text{cond}(A, \text{Inf}) = \text{Inf}$

(e) $p(2+y10^{-2})$

$= c_0 + (2+y10^{-2})c_1 + (2+y10^{-2})^2 c_2$

and each term is a polynomial of degree ≤ 2. Similarly $q(2+y10^{-6})$ is a polynomial of degree ≤ 2.

Hence $f(y)$ is also a polynomial of degree ≤ 2.

$f(0) = p(2) - q(2) = \sqrt{2} - \sqrt{2} = 0$

$f(1) = p(2+10^{-2}) - q(2+10^{-6}) = \sqrt{3} - \sqrt{3} = 0$

Similarly $f(2) = \sqrt{5} - \sqrt{5} = 0$.

(f) Since f is polynomial of degree 2 and f has ~~thr~~ three distinct roots $(y=0,1,2)$ we have $f(y)=0$ for all y (i.e. f is the zero polynomial). Hence $p(2+10^{-2}y) = q(2+10^{-6}y)$ for all y,

$$\alpha_2 = p(2+10^{-2}(\tfrac{1}{2})) = q(2+10^{-6}(\tfrac{1}{2})) = \alpha_6$$

(g) Similarly

$$g(y) = q(2+10^{-6}y) - r(2+10^{-7}y)$$

is of degree ≤ 2 with roots $y = 0, 1, 2$,

so
$$q(2+10^{-6}y) = r(2+10^{-7}y)$$

is an equality of polynomials.

Hence

$$\alpha_6 = q(2+10^{-6}(\tfrac{1}{2})) = r(2+10^{-7}(\tfrac{1}{2}))$$

$$= \alpha_7$$

[And for any $n \in \mathbb{Z}$ we can define α_n, or even for any $t \in \mathbb{R}$ we can define α_t, and all these α_t are independent of t.]

(4) (a) $\|A\|_\infty = 3+\varepsilon$ (from the bottom row) $\|A^{-1}\|_\infty = \frac{4+\varepsilon}{\varepsilon}$ (from the top row)

(b) Since the bottom row of A has entries of same sign,

(*) $\left\| A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty = \|A\|_\infty \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty$

(see Homework 6); similarly, since the top row of A^{-1} has entries of opposite sign, we have (Homework 6)

$$\left\| A^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\|_\infty = \|A^{-1}\|_\infty \left\| \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\|_\infty .$$

Moreover

$$\left\| A^{-1} \begin{bmatrix} \delta \\ -\delta \end{bmatrix} \right\|_\infty = \left\| \delta A^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\|_\infty = |\delta| \left\| A^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\|_\infty$$

and $$\left\| \begin{bmatrix} \delta \\ \delta \end{bmatrix} \right\|_\infty = |\delta| \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty$$

and combining these with $(*)$ we get

$$\left\| A^{-1} \begin{bmatrix} \delta \\ -\delta \end{bmatrix} \right\|_\infty = \| A^{-1} \|_\infty \left\| \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\|_\infty$$

[Of course, you can alternatively do the direct calculation.]

(c) The previous part implies that (eq 3) is satisfied for

$$\vec{b}_{error} = \begin{bmatrix} \delta \\ -\delta \end{bmatrix}, \quad \vec{x}_{true} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and (eq 4) gives

$$\vec{x}_{error} = A^{-1} \vec{b}_{error}$$

$$= \frac{1}{\varepsilon} \begin{bmatrix} 2+\varepsilon & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \delta \\ -\delta \end{bmatrix}$$

$$= \frac{1}{\varepsilon} \begin{bmatrix} 2+\varepsilon & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \delta$$

$$= \frac{1}{\varepsilon}\begin{bmatrix} 4+\varepsilon \\ -2 \end{bmatrix}\delta$$

and (eq 5) gives

$$\vec{X}_{approx} = \vec{X}_{true} + \vec{X}_{error}$$

So

$$\vec{X}_{approx}(\delta) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \delta\begin{bmatrix} 4+\varepsilon \\ -2 \end{bmatrix}\frac{1}{\varepsilon}$$

(d) $\vec{X}_{approx}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{X}_{true}$ in (c)

(5)(a) $\text{RelError}_\infty\left(X_{simpler}(\eta), X_{simpler}(0)\right)$

$$= \frac{\left\| \begin{bmatrix} 2 \\ -1 \end{bmatrix} \eta \right\|_\infty}{\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty} = \frac{2|\eta|}{1},$$

$$\mathrm{RelError}_\infty\left(A\,x_{simpler}(\eta),\ A\,x_{simpler}(0)\right)$$

$$= \frac{\left\| A\begin{bmatrix}2\\-1\end{bmatrix}\eta \right\|_\infty}{\left\| A\begin{bmatrix}1\\1\end{bmatrix}\right\|_\infty} = \frac{\left\| \begin{bmatrix}1 & 2\\1 & 2+\varepsilon\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}\eta \right\|_\infty}{\left\| \begin{bmatrix}1 & 2\\1 & 2+\varepsilon\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} \right\|_\infty}$$

$$= \frac{\left\| \begin{bmatrix}0\\-\varepsilon\end{bmatrix}\eta \right\|_\infty}{\left\| \begin{bmatrix}3\\3+\varepsilon\end{bmatrix} \right\|} = \frac{\varepsilon|\eta|}{3+\varepsilon}$$

Hence the ratio of these two expressions equals

$$2|\eta| \Big/ \left(\frac{\varepsilon|\eta|}{3+\varepsilon}\right) = \frac{6+2\varepsilon}{\varepsilon} \geq \frac{6}{\varepsilon}.$$

(b) $x_{approx}(\delta) - x(2\delta/\varepsilon)$

$$= \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}4+\varepsilon\\-2\end{bmatrix}\frac{\delta}{\varepsilon} - \left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\-1\end{bmatrix}\frac{2\delta}{\varepsilon}\right)$$

$$= \begin{bmatrix} 4+\varepsilon \\ -2 \end{bmatrix} \frac{\delta}{\varepsilon} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} \left(\frac{2\delta}{\varepsilon} \right)$$

$$= \left(\begin{bmatrix} 4+\varepsilon \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right) \left(\frac{\delta}{\varepsilon} \right)$$

$$= \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} \left(\frac{\delta}{\varepsilon} \right) = \begin{bmatrix} \delta \\ 0 \end{bmatrix}$$

Hence

$$\| x_{approx}(\delta) - x(2\delta/\varepsilon) \|_\infty = |\delta|$$

Also $\| x(\eta) \|_\infty = \left\| \begin{bmatrix} 1+2\eta \\ 1-\eta \end{bmatrix} \right\|$

$$= \max \left(|1+2\eta|, |1-\eta| \right)$$

$$\geq \begin{cases} 1+2\eta & \text{if } \eta \geq 0 \\ 1+\eta & \text{if } \eta < 0 \end{cases}$$

$\geq 1 \quad \text{for all } \eta$

Hence

$$\frac{\| x_{approx}(\delta) - x(2\delta/\varepsilon) \|_\infty}{\| x(2\delta/\varepsilon) \|_\infty}$$

$$\leq \frac{|\delta|}{1} = |\delta| .$$

6 (a)

$$A\,\overrightarrow{c_{\text{approx}}}(\eta) = \begin{bmatrix} 1 & 2 \\ 1 & 2+\varepsilon \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \eta \right)$$

$$= \begin{bmatrix} 3 \\ 3+\varepsilon \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \eta = \begin{bmatrix} 3 \\ 3+\varepsilon+\varepsilon\eta \end{bmatrix}$$

Hence $\quad A\,\vec{c} = (\text{the above with } \eta = 0)$

$$= \begin{bmatrix} 3 \\ 3+\varepsilon \end{bmatrix}$$

$$B\,\overrightarrow{c_{\text{approx}}}(\eta)$$

$$= \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \longmapsto \begin{bmatrix} y_0 \\ (y_1 - y_0)/\varepsilon \end{bmatrix} \text{ applied to } A\,\overrightarrow{c_{\text{approx}}}(\eta)$$

$$= \begin{bmatrix} 3 \\ (3+\varepsilon+\varepsilon\eta-3)/\varepsilon \end{bmatrix} = \begin{bmatrix} 3 \\ 1+\eta \end{bmatrix}$$

and

$$B\,\vec{c} = (\text{the above with } \eta = 0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

So:

$$\text{RelError}_\infty(A\vec{c}_{approx}, Ac)$$

$$= \left\| \begin{bmatrix} 0 \\ \varepsilon\eta \end{bmatrix} \right\|_\infty \Big/ \left\| \begin{bmatrix} 3 \\ 3+\varepsilon \end{bmatrix} \right\|_\infty$$

$$= \frac{|\varepsilon\eta|}{3+\varepsilon} = \frac{\varepsilon|\eta|}{3+\varepsilon}$$

$$\text{RelError}_\infty(B\vec{c}_{approx}, B\vec{c}) =$$

$$\left\| \begin{bmatrix} 0 \\ \eta \end{bmatrix} \right\|_\infty \Big/ \left\| \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\|_\infty = |\eta|/3$$

So

$$\underline{9 \cdot \text{RelError}_\infty(B\vec{c}_{approx}, B\vec{c})} = 9 \cdot \frac{|\eta|}{3} = \underline{3|\eta|}$$

$$\underline{\frac{6+2\varepsilon}{\varepsilon} \cdot \text{RelError}_\infty(A\vec{c}_{approx}, Ac)}$$

$$= \left(\frac{6+2\varepsilon}{\varepsilon}\right)\left(\frac{\varepsilon|\eta|}{3+\varepsilon}\right) = \frac{6+2\varepsilon}{3+\varepsilon}|\eta| = \underline{2|\eta|}$$

Since $3|\eta| \geq 2|\eta|$,

(17) holds.

(b) Instead of (15) we have

$$\mathrm{RelError}_\infty\left(\overrightarrow{x_{approx}}(\delta), \overrightarrow{x_{approx}}(0)\right)$$

$$= \frac{12+7\varepsilon+\varepsilon^2}{\varepsilon} \mathrm{RelError}_\infty\left(A\overrightarrow{x_{approx}}(\delta), A\overrightarrow{x_{approx}}(0)\right)$$

we get (17) with $\frac{6+2\varepsilon}{\varepsilon}$ replaced with $\frac{12+7\varepsilon+\varepsilon^2}{\varepsilon}$

and $\vec{x}(\eta)$ replaced with $\overrightarrow{x_{approx}}(\delta)$, i.e.

$$\boxed{9 \; \mathrm{RelError}_\infty\left(\overrightarrow{x_{approx}}(\delta), \overrightarrow{x_{approx}}(0)\right) \geq \frac{12+7\varepsilon+\varepsilon^2}{\varepsilon} \mathrm{RelError}_\infty\left(A\overrightarrow{x_{approx}}(\delta), A\overrightarrow{x_{approx}}(0)\right)}$$

[Since $\overrightarrow{x_{approx}}(\delta)$ is a more complicated expression than $\vec{x}(\eta)$, I did not ask you to explicitly verify the inequality above.]