Homework 6 Solutions, CPSC 303, 2024

(2) (a) Subtract Row 1 from Rows 2 and 3

gives:

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & \varepsilon & 4\varepsilon+\varepsilon^2 \\ 0 & -\varepsilon & -4\varepsilon+\varepsilon^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 - y_0 \\ y_2 - y_0 \end{bmatrix}$$

adding row 2 to row 3 gives

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & \varepsilon & 4\varepsilon+\varepsilon^2 \\ 0 & 0 & 2\varepsilon^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 - y_0 \\ y_2 - y_0 + (y_1 - y_0) \end{bmatrix}$$

So

$$2\varepsilon^2 c_2 = y_2 + y_1 - 2y_0$$

So

$$c_2 = \frac{y_2 + y_1 - 2y_0}{2\varepsilon^2}$$

(b) So

$$c_2(\varepsilon) = \frac{f(2-\varepsilon) + f(2+\varepsilon) - 2f(2)}{2\varepsilon^2}$$

taking $\varepsilon \to 0$, using L'Hôpital's Rule we differentiate the numerator and denominator separately and find:

$$\lim_{\varepsilon \to 0} C_2(\varepsilon) = \lim_{\varepsilon \to 0} \frac{-f'(2-\varepsilon) + f'(2+\varepsilon) - 0}{4\varepsilon}$$

and other application of L'Hôpital's rule gives that this limit

$$= \lim_{\varepsilon \to 0} \frac{f''(2-\varepsilon) + f''(2+\varepsilon)}{4} = \frac{2f''(2)}{4} = \frac{f''(2)}{2}$$

Alternative: By Talyor's theorem we have

$$f(2+\varepsilon) = f(2) + \varepsilon f'(2) + \frac{\varepsilon^2}{2} f''(2) + O(\varepsilon^3)$$

$$f(2-\varepsilon) = f(2) - \varepsilon f'(2) + \frac{\varepsilon^2}{2} f''(2) + O(\varepsilon^3)$$

So

$$f(2+\varepsilon) + f(2-\varepsilon) = 2f(2) + \varepsilon^2 f''(2) + O(\varepsilon^3)$$

So

$$f(2+\varepsilon) + f(2-\varepsilon) - 2f(2) = \varepsilon^2 f''(2) + O(\varepsilon^3)$$

Hence

$$\frac{f(2+\varepsilon)+f(2-\varepsilon)-2f(2)}{2\varepsilon^2} = \frac{f''(2)}{2} + O(\varepsilon)$$

so

$$\lim_{\varepsilon\to 0} C_2(\varepsilon) = \lim_{\varepsilon\to 0} \frac{f(2+\varepsilon)+f(2-\varepsilon)-2f(2)}{2\varepsilon^2}$$

$$= \frac{f''(2)}{2}.$$

3(a) $|ax_1+bx_2| \leq |ax_1| + |bx_2| \leq |am| + |bm|$

$$\leq m(|a|+|b|)$$

3(b) Similarly $|cx_1+dx_2| \leq m(|c|+|d|)$,

so

$$\|Ax\|_\infty = \max(|ax_1+bx_2|, |cx_1+dx_2|)$$

$$\leq m \max(|a|+|b|, |c|+|d|)$$

$$= \|\vec{x}\|_\infty \max(|a|+|b|, |c|+|d|)$$

3(c) $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a+b\\c+d\end{bmatrix}$, so if a,b are both positive or zero or negative we have $|a+b| = |a|+|b|$, so

$$\left\| A\begin{bmatrix}1\\1\end{bmatrix} \right\|_\infty \geq |a+b| = |a|+|b|.$$

If a,b have opposite signs, then $|a-b| = |a|+|b|$, and

$$\left\| A\begin{bmatrix}1\\-1\end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix}a-b\\c-d\end{bmatrix} \right\|_\infty \geq |a|+|b|.$$

(d) Similarly we have

$$\left\| A\begin{bmatrix}1\\1\end{bmatrix} \right\|_\infty \text{ or } \left\| A\begin{bmatrix}1\\-1\end{bmatrix} \right\|_\infty \geq |c|+|d|.$$

Since $\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|_\infty = \left\|\begin{bmatrix}1\\-1\end{bmatrix}\right\|_\infty = 1$, it follows that for $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$ or $\begin{bmatrix}1\\-1\end{bmatrix}$ we have

$$\| A\vec{x} \|_\infty \geq \max(|a|+|b|, |c|+|d|)\, \|\vec{x}\|_\infty .$$

So this and 3(b) implies

$$\|A\|_\infty \overset{\text{def}}{=} \max_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty}$$

$$= \max\left(|a|+|b|, |c|+|d| \right)$$

4. We have

$$\left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\|_1 = \left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \right\|_\infty$$

$$= \left\| \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\|_\infty$$

$$= \max\left(|a|+|c|, |b|+|d| \right)$$

(which agrees with the formula in Section 6 of the handout).

5 (a) Since $x_0 = 2$, $x_1 = 2+\varepsilon$, $x_2 = 2-\varepsilon$,

the bottom right entry of A^{-1}

is

$$\frac{1}{(x_2-x_0)}\frac{1}{(x_1-x_0)} = \frac{1}{(-\varepsilon)}\frac{1}{(-2\varepsilon)} = \frac{1}{2\varepsilon^2}$$

Hence

$$\|A^{-1}\|_\infty \geq \frac{1}{2\varepsilon^2}.$$

(b) We have for $A = A(\varepsilon)$,

$$A\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \implies \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = A^{-1}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

(by multiplying both sides by A^{-1} on the left)

hence

$$c_2 = \left[\text{bottom row of } A^{-1}\right]\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

Since

$$C_2 = \frac{y_1 + y_2 - 2y_0}{2\varepsilon^2} = \frac{1}{2\varepsilon^2}\begin{bmatrix} -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

we have

$$\begin{bmatrix} \text{bottom row of } A^{-1} \end{bmatrix} = \frac{1}{2\varepsilon^2}\begin{bmatrix} -2 & 1 & 1 \end{bmatrix}$$

So, indeed, the bottom right entry of

A^{-1} is $\frac{1}{2\varepsilon^2} \cdot 1 = \frac{1}{2\varepsilon^2}$.

(c) Since A has an entry equal to 4,

we have $\|A\|_\infty \geq 4$, and hence

$$\|A(\varepsilon)\|_\infty \|A^{-1}(\varepsilon)\|_\infty \geq 4 \cdot \frac{1}{2\varepsilon^2} \geq 2\varepsilon^2.$$

Note: we can get a better estimate

using the fact that

$$A = \begin{bmatrix} 1 & 2 & 4 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \text{ so } \|A\|_\infty \geq 1+2+4=7$$

and that

$$A^{-1}(\varepsilon) = \begin{bmatrix} \vdots & \vdots & \vdots \\ -1/\varepsilon^2 & 1/2\varepsilon^2 & 1/2\varepsilon^2 \end{bmatrix}$$

to see that

$$\|A^{-1}(\varepsilon)\|_\infty \geq \left|\frac{-1}{\varepsilon^2}\right| + \left|\frac{1}{2\varepsilon^2}\right| + \left|\frac{1}{2\varepsilon^2}\right| = \frac{2}{\varepsilon^2}.$$

Hence

$$\|A(\varepsilon)\|_\infty \|A^{-1}(\varepsilon)\|_\infty \geq 7 \cdot \frac{2}{\varepsilon^2} = \frac{14}{\varepsilon^2}.$$

In any event

$$K_\infty(A) = \|A(\varepsilon)\|_\infty \|A^{-1}(\varepsilon)\|_\infty$$

tends to ∞ as $\varepsilon \to 0$, at least as fast as $14/\varepsilon^2$ tends to ∞.