CPSC 303, Homework Solutions 5, 2024

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(2) (a) $\frac{d}{dt}\left(\vec{x}\cdot\vec{z}\right) = \frac{d}{dt}\left(x_1 z_1 + x_2 z_2\right)$

$= \frac{d}{dt}(x_1 z_1) + \frac{d}{dt}(x_2 z_2)$

(product rule)

$\Longrightarrow (x_1 \dot{z}_1 + \dot{x}_1 z_1) + (x_2 \dot{z}_2 + \dot{x}_2 z_2)$

$= \vec{x}\cdot\dot{\vec{z}} + \dot{\vec{x}}\cdot\vec{z}$

(b) $\frac{d}{dt}\left(\|\vec{x}\|^2\right) = \frac{d}{dt}\left(\vec{x}\cdot\vec{x}\right)$,

using part (a) with $\vec{z} = x$,

$= \vec{x}\cdot\dot{\vec{x}} + \dot{\vec{x}}\cdot\vec{x} = 2\,\vec{x}\cdot\dot{\vec{x}}$

(c) Since m is constant, using part (b)

with $\dot{\vec{x}}$ replacing $\vec{x}$:

$$\frac{d}{dt}\left(m\|\dot{\vec{x}}\|^2\right) = m\frac{d}{dt}\left(\|\dot{\vec{x}}\|^2\right)$$

$$= m\,2\,\dot{\vec{x}}\cdot\ddot{\vec{x}}$$

(d) $$\frac{d}{dt}\|x\| = \frac{d}{dt}\sqrt{\vec{x}\cdot\vec{x}} \overset{\text{chain rule}}{=\!=}$$

$$\frac{1}{2}\frac{1}{\sqrt{\vec{x}\cdot\vec{x}}}\frac{d}{dt}\left(\vec{x}\cdot\vec{x}\right) = \frac{1}{2}\frac{1}{\|\vec{x}\|^2}\,2\vec{x}\cdot\dot{\vec{x}}$$

$$= \frac{1}{\|\vec{x}\|^2}\,\vec{x}\cdot\dot{\vec{x}}$$

(e) $$\frac{d}{dt}U(\|\vec{z}\|) \overset{\text{chain rule}}{=\!=} U'(\|\vec{z}\|)\frac{d}{dt}\|\vec{z}\|$$

$$\overset{\text{part(d)}}{=\!=} u(\|\vec{z}\|)\frac{1}{\|\vec{z}\|^2}\,\vec{z}\cdot\dot{\vec{z}}$$

(f) $$\frac{d}{dt}\frac{1}{2}m\|\dot{x}\|^2 \overset{\text{part(c)}}{=\!=} m\,\dot{\vec{x}}\cdot\ddot{\vec{x}}$$

central force $\quad m \dot{\vec{x}} \cdot \left(- u(\|\vec{x}\|) \frac{\vec{x}}{\|\vec{x}\|} \right)$

$$= -m\, u(\|\vec{x}\|) \frac{\vec{x} \cdot \dot{\vec{x}}}{\|\vec{x}\|}$$

$$\frac{d}{dt}\left(m\, U(\|\vec{x}\|) \right) \quad \text{part (e) with } \vec{z} \to \vec{x}$$

$$= m\, u(\|\vec{x}\|) \frac{\vec{x} \cdot \dot{\vec{x}}}{\|\vec{x}\|}$$

Hence

$$\frac{d}{dt} \text{Energy}(t) = \frac{d}{dt}\left(\frac{1}{2} m \|\dot{\vec{x}}\|^2 \right) + \frac{d}{dt}\left(m\, U(\|\vec{x}\|) \right)$$

$$= 0$$

2. (a) (i)

$E(y_0) = -0.68$, $E(y_N) = -0.0175$

$E(y_i)$ appears to strictly increase (i.e. decrease in absolute value), especially when the planet is near $(0,0)$, for $t \leq$ roughly 50. [If you examine the energy differences closely, the first decrease (increase in absolute value) occurs from y_{469} to y_{470}.]

From then on, there is a slight decrease in energy.

The ellipse-like orbits (which are ellipses in exact computation) seem to get larger as time increases [which is consistent with a negative but increasing energy}.

(ii) $E(y_0) = -0.68$, $E(y_N) = -0.2137$,

and the same happens as in (i),

except the enery always increases

(like (i), the increase is faster the

closer the planet gets to $\vec{O} = (0,0)$).

(iii) $E(y_0) = -0.68$, $E(y_N) = -0.4980$,

and similar to (ii).

(b) $E(y_0) = -0.68$, $E(y_N) = -0.4071$.

Here the energy decreases (increases

in absolute value) when the planet

moves toward the sun, and

the reverse as it moves away,

with an overall increase in energy at each revolution. (Hence) the elliptic shape of the orbit slightly increases.

[This wasn't assigned, but if you take $h = 0.01$, $N = 6000$ in the explicit trapezoidal method, $E[y_i]$ remains close to $E[y_0] = -0.68$ for all i (between $-0.68036\ldots$ and $-0.67935\ldots$).]

(4) (a) This is $(\sigma^2 - 1)(x_n) = 0$,

solving $r^2 - 1 = 0$ we get $r = \pm 1$.

Hence $x_n = c_1(1)^n + c_2(-1)^n = c_1 + c_2(-1)^n$.

(b) $x_0 = c_1 + c_2(-1)^0 = c_1 + c_2 = 5$

$x_1 = c_1 - c_2 = 7$

adding these two equation gives $2c_1 = 12$

so $c_1 = 6$, and then $c_2 = 5 - c_1 = 5 - 6 = -1$.

Hence $x_n = 6 - (-1)^n$.

(c) We get $x_9 = 6 - (-1)^9 = 7$,

$x_{10} = 6 - (-1)^{10} = 5$, and similarly $x_{11} = 7$, $x_{12} = 5$.

This is not the fastest way, since

$$x_{n+2} - x_n = 0 \Rightarrow x_n = x_{n+2}$$

and hence

$$x_0 = x_2 = x_4 = \cdots \quad \text{and} \quad x_1 = x_3 = x_5 = \cdots$$

and hence, for $x_0 = 5$, $x_1 = 7$ we have

$$5 = x_0 = x_2 = x_4 = \cdots \quad \text{and} \quad 7 = x_1 = x_3 = x_5 = \cdots$$

(d)

$$Y_{n+1} = \begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Y_n$$

Hence $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$,

So

$$I = A^0 = A^2 = A^4 = \cdots$$

and

$$A = A^3 = A^5 = \cdots = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

So

$$A^9 = A^{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A^{10} = A^{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } I$$

(5) (a) $x_0 = c_1 2^0 + c_2 \cdot 0 \cdot 2^0 = c_1$

$x_1 = c_1 2^1 + c_2 \cdot 1 \cdot 2^1 = 2(c_1+c_2)$

So

$c_1 = x_0, \quad c_2 = \frac{x_1}{2} - c_1 = \frac{x_1 - 2x_0}{2}$ OR $\frac{x_1}{2} - x_0$

Hence

$$x_n = x_0 2^n + \left(\frac{x_1}{2} - x_0\right) n 2^n$$

OR, OPTIONALLY,

$$= x_0 (2^n)(1-n) + x_1 \frac{n 2^n}{2}$$

OR

$$x_0 (1-n) 2^n + x_1 n 2^{n-1}.$$

(b) $$\vec{y}_{n+1} = \begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} \vec{y}_n .$$

Hence

$$\vec{Y}_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A^n \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}, \text{ where } A = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$$

Since

$$2I - \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} = N$$

and we've seen in class that $N^2 = 0$,

we have $2I - A = N$ so $A = 2I - N$,

so

$$A^n = (2I)^n + \binom{n}{1}(2I)^{n-1}(-N) + \text{terms with}$$

N^2 or N^3 or ...

$$= 2^n I + n 2^{n-1}(-N)$$

$$= \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix} + n 2^{n-1} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2^n + n2^n & -n2^{n+1} \\ n2^{n-1} & 2^n - n2^n \end{bmatrix}$$

So

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} \text{whatever} & \text{whatever} \\ n2^{n-1} & 2^n(1-n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$$

$\Rightarrow$

$$x_n = n2^{n-1} x_1 + 2^n(1-n) x_0$$

(which argees with (a)).

(c)

$$(A(\varepsilon))^n = S(\varepsilon) \begin{bmatrix} 2^n & 0 \\ 0 & (2+\varepsilon)^n \end{bmatrix} (S(\varepsilon))^{-1}$$

where

$$S(\varepsilon) = \underbrace{\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}}_{M_1} + \varepsilon \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{M_2}$$

$$\begin{bmatrix} 2^n & 0 \\ 0 & (2+\varepsilon)^n \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 0 & 2^n + n2^{n-1}\varepsilon + O(\varepsilon^2) \end{bmatrix}$$

$$= \underbrace{2^n I}_{M_3} + \varepsilon \underbrace{n 2^{n-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{M_4}$$

$$(S(\varepsilon))^{-1} = \frac{-1}{\varepsilon} \begin{bmatrix} 1 & -2-\varepsilon \\ -1 & 2 \end{bmatrix}$$

$$= \frac{-1}{\varepsilon} \left(\underbrace{\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}}_{M_5} + \varepsilon \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}_{M_6} \right)$$

Since $(M_1 + \varepsilon M_2)(M_3 + \varepsilon M_4)(M_5 + \varepsilon M_6)$

$$= m_1 m_3 m_5 +$$

$$\varepsilon \left[m_2 m_3 m_5 + m_1 m_4 m_5 + m_1 m_3 m_6 \right]$$

We calculate

$$m_1 m_3 m_5 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} 2^n I \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} 2^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(which has to be the case if $\lim_{\varepsilon \to 0} (A(\varepsilon))^n$ has a finite limit).

(1) $$m_2 m_3 m_5 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} 2^n \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = 2^n \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$

(2) $$m_1 m_4 m_5 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} n 2^{n-1} \begin{bmatrix} 0 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

$$= n 2^{n-1} \begin{bmatrix} a & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = n 2^{n-1} \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

(3) $M_1 M_3 M_6 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} 2^n \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$

$$= 2^n \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix}$$

And the sum of (1), (2), (3) is

$$2^n \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} + n 2^{n-1} \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} + 2^n \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix}$$

$$= 2^n \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + n 2^{n-1} \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

Hence

$$(A(\varepsilon))^n = \frac{-1}{\varepsilon} \left(\varepsilon \left(2^n \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + n 2^{n-1} \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \right) + O(\varepsilon^2) \right)$$

$$= 2^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + n 2^{n-1} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} + O(\varepsilon)$$

So

$$\lim_{\varepsilon \to 0} (A(\varepsilon))^n = 2^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + n2^{n-1} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}.$$

$$= \begin{bmatrix} 2^n + n2^n & -n2^{n+1} \\ n2^{n-1} & 2^n - n2^n \end{bmatrix}$$

(which does agree with our answer in part (b)).