CPSC 303, 2024. Homework Solutions 4

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(2)(a) $\psi_i = y_i + h y_i = (1+h) y_i$

$$y_{i+1} = y_i + h \frac{y_i + \psi_i}{2}$$

$$= y_i + h \frac{y_i + (1+h) y_i}{2}$$

$$= y_i \left(1 + h + \frac{h^2}{2} \right)$$

(b) $y_N = \left(1+h+\frac{h^2}{2}\right) y_{N-1} =$

$$\left(1+h+\frac{h^2}{2}\right)^2 y_{N-1} = \cdots$$

$$= \left(1+h+\frac{h^2}{2}\right)^N y_0 = \left(1+h+\frac{h^2}{2}\right)^N$$

Since $h = \frac{1}{N}$, we have

$$y_{N,\text{trap}} = \left(1 + \frac{1}{N} + \frac{1}{2N^2}\right)^N$$

(c) Euler's method gives

$$y_{i+1} = y_i + h y_i = (1+h),$$

so similarly $y_{N,\text{Eul}} = (1+h)^N y_0 = \left(1 + \frac{1}{N}\right)^N$

(d)

$$\ln(y_{N,\text{Eul}}) = N \ln\left(1 + \frac{1}{N}\right)$$

$$= N\left(\frac{1}{N} - \frac{1}{N^2}/2 + O\left(\frac{1}{N^3}\right)\right)$$

$$= 1 - \frac{1}{2N} + O\left(\frac{1}{N^2}\right)$$

but

$$\ln(y_{N,trap}) = N \ln\left(1 + \frac{1}{N} + \frac{1}{2N^2}\right)$$

$$= N\left(\frac{1}{N} + \frac{1}{2N^2} - \left(\frac{1}{N} + \frac{1}{2N^3}\right)^2 / 2 \right.$$

$$\left. O\left(\frac{1}{N} + \frac{1}{2N^3}\right)^3 \right)$$

$$= N\left(\frac{1}{N} + \frac{1}{2N^2} - \frac{1}{2N^2} + O\left(\frac{1}{N^3}\right)\right)$$

$$= N\left(\frac{1}{N} + O\left(\frac{1}{N^3}\right)\right) = 1 + O\left(\frac{1}{N^2}\right).$$

So $\ln(y_{N,Euler}) = 1 - \frac{1}{2N} + O\left(\frac{1}{N^2}\right)$

$\ln(y_{N,trap}) = 1 + O\left(\frac{1}{N^2}\right)$

and $\ln(\underbrace{y(1)}_{\text{the exact answer}}) = 1$

Hence

$$\ln(y(1)) - \ln(y_{N,trap}) = O\left(\frac{1}{N^2}\right)$$

while

$$\ln(y(1)) - \ln(y_{N,Eul}) = \frac{1}{2N} + O\left(\frac{1}{N^2}\right)$$

So $\ln(y(1)) - \ln(y_{N,trap})$

is smaller as $N \to \infty$

(3a) If $y(t) = at^2 + bt + c$, then

$$y' - 2y = (2at + b) - 2(at^2 + bt + c)$$

$$= (-2a)t^2 + (2a - 2b)t + b - 2c$$

So if this equals t^2, we

have $\left.\begin{aligned} -2a &= 1 \\ 2a - 2b &= 0 \\ b - c &= 0 \end{aligned}\right\}$

so: $a = -1/2$, $b = a = -1/2$, $c = \frac{b}{2} = \frac{-1}{4}$

So $y(t) = \frac{-1}{2} t^2 - \frac{1}{2} t - \frac{1}{4}$

(3b) $(y+z)' - 2(y+z)$

$= (y' - 2y) + (z' - 2z)$

$= \quad t^2 \quad + 0 \quad = t^2$

(3c) If $y' - 2y = t^2 = (y+z)' - 2(y+z)$

then $\big((y+z)' - 2(y+z)\big) - (y' - 2y) = 0$

so $(y+z-y)' - 2(y+z-y) = 0$

So $\quad z' - 2z = 0$

(3d) Since $z' - 2z$ has the general solution (of $z' = 2z$ and hence) $z(t) = Ce^{2t}$, and $y(t) = -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{1}{4}$ has $y' - 2y = t^2$, we have

$$(y+z)' - 2(y+z) = t^2$$

iff

$$y + z = \left(-\frac{1}{2}t^2 - \frac{1}{2}t - \frac{1}{4}\right) + Ce^{2t}$$

(3e) If $\quad y(t) =$ [illegible]

and $y(t_0) = y_0$, then

$$C e^{2t_0} = \frac{1}{2} t_0^2 + \frac{1}{2} t_0 + \frac{1}{4}$$

So C is uniqule given as

$$C = \left(\frac{1}{2} t_0^2 + \frac{1}{2} t_0 + \frac{1}{4} \right) e^{-2t_0}.$$

(4a) If $x_n = an^2 + bn + c$

then

$$x_{n+1} - 2x_n =$$

$$a\left((n+1)^2 - 2n^2 \right) + b\left((n+1) - 2n \right)$$

$$+ c(1-2)$$

$$= a(-n^2 + 2n + 1) + b(-n+1) - c$$

$= n^2(-a) + n(2a-b) + a+b-c$

So this equals n^2 iff

$$\left.\begin{aligned} -a &= 1 \\ 2a-b &= 0 \\ a+b-c &= 0 \end{aligned}\right\} \text{ so } \begin{aligned} &a=-1,\ b=2a=-2 \\ &c=a+b=-3 \end{aligned}$$

So $x_n = -n^2 - 2n - 3$

(4b) $x_{n+1} = 2x_n$ so

$x_n = c\,2^n$

(4c) Similarly, the general solution to $x_{n+1} - 2x_n = n^2$ is equal to any particular

solution, plus any solution
to the "homogeneous form
of this equation" (namely $x_{n+1} - 2x_n = 0$),
Hence the general solution is

$$x_n = -n^2 - 2n - 3 + C\,2^n.$$

(5a) MATLAB gives $(1/2)^{1074}$ as
$4.94065\ldots \times 10^{-324}$ and $(1/2)^{1075}$ as 0.

(5b) $r^2 - (3/2)r + (1/2) = 0$ gives $r = 1, 1/2$,
so the general solution is

$$x_n = c_1 1^n + c_2 (1/2)^n = c_1 + c_2 (1/2)^n$$

(5c) $x_1 = c_1 + c_2(1/2) = 1$

$x_2 = c_1 + c_2(1/2)^2 = 1/2$

so $c_1 = 0$, $c_2 = 2$, so

$x_n = 2(1/2)^n$

(5d) MATLAB reports:

$x\{100\} \cdot 2^{99} = 1$, $\quad x\{100\} \cdot 2^{99} - 1 = 0$

(5e) (i) No: x_n for $n \leq 1200$ is never reported as 0.

(ii) An examination of

for n = 1:1200, x{n} - (1/2)^(n-1), end

(there are many variants) shows that

$x\{1075\}$ is reported as $(1/2)^{1074}$

(and both as $4.9406\ldots \times 10^{-324}$) while

$x\{1076\}$ is reported as the same (!)

while $(1/2)^{1075}$ (as above) is reported as 0.

(iii) We take $n_0 = 1076$ and note:

$(3/2)\, x\{1075\}$ reported as

$9.88131\ldots \times 10^{-324}$ (twice as large

as $x\{1076\}$) and

$(1/2)\, x\{1074\}$ is reported (correctly)

as $4.9406\ldots \times 10^{-324}$

(iv) By multiples of $m = (1/2)^{1074}$

$= 4.9406\ldots \times 10^{-324}$, MATLAB reports

$$\left\{ \begin{array}{ll} x\{1072\} = 8m, & x\{1073\} = 4m \\ x\{1074\} = 2m, & x\{1075\} = m \end{array} \right\}$$

and

$x\{1076\} = m, \quad x\{1077\} = 2m,$

$x\{1078\} = 3m, \quad x\{1079\} = 3m,$

$x\{1080\} = 2m, \quad x\{1081\} = m$

$x\{1082\} = m, \quad x\{1083\} = 2m$

$\vdots$

And the pattern repeats every 6 steps.

[Since this is a 3-term recurrence, we know that since

$$x\{1076\} = x\{1082\} = m$$

$$x\{1077\} = x\{1083\} = 2m$$

that things have to cycle in this pattern.]