CPSC 303, 2024 Homework 3 Solutions

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(2) $$e^{(A+B)h} = I + (A+B)h + \frac{((A+B)h)^2}{2} + O(h^3)$$

$$= I + Ah + Bh + \frac{h^2}{2}(A+B)(A+B) + O(h^3)$$

and

$$(A+B)(A+B) = (A+B)A + (A+B)B$$

$$= A^2 + BA + AB + B^2$$

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$$e^{Ah}e^{Bh} =$$

$$\left(I + Ah + \frac{A^2h^2}{2} + O(h^3)\right)\left(I + Bh + \frac{B^2h^2}{2} + O(h^3)\right)$$

$$= \left(I + Ah + \frac{A^2h^2}{2}\right) + \left(I + Ah + \frac{A^2h^2}{2}\right)Bh$$

$$+ \left(I + Ah + \frac{A^2h^2}{2}\right)\frac{B^2h^2}{2} + O(h^3)$$

$$= I + Ah + \frac{A^2h^2}{2} + Bh + ABh^2 + O(h^3)$$

$$+ \frac{B^2h^2}{2} + O(h^3)$$

$$= I + (A+B)h + \frac{h^2}{2}\left(A^2 + 2AB + B^2\right) + O(h^3)$$

So, using the above (and noting the common term $I + (A+B)h$), we have

$$e^{(A+B)h} - e^{Ah}e^{Bh} =$$

$$\frac{h^2}{2}\left(\left(A^2 + BA + AB + B^2\right) - \left(A^2 + 2AB + B^2\right)\right) + O(h^3)$$

$$= \frac{h^2}{2}\left(BA - BA\right) + O(h^3)$$

(3) (a) $0 = y_i + h|y_i|^{1/2}$, so

$y_i = -h|y_i|^{1/2}$ which is $\leq 0$

So $y_i = -|y_i|$ and hence $|y_i| = h|y_i|^{1/2}$

so either $|y_i| = 0$ or we divide by $|y_i|^{1/2}$

to get $|y_i|^{1/2} = h$ so $|y_i| = h^2$

so (since $y_i \leq 0$) $y_i = -h^2$.

Hence either $|y_i| = 0$ (so $y_i = 0$)

or $y_i = -h^2$.

(3 b) (i) Since $y_{i+1} < 0$ and

$y_{i+1} = y_i + h|y_i|^{1/2}$, we have

$y_i = y_{i+1} - h|y_i|^{1/2} \leq y_{i+1} < 0$.

(ii) Setting $x = \sqrt{-y_i}$, which is therefore $>0$, we have $x^2 = -y_i$, so

$$y_i = y_{i+1} - h|y_i|^{1/2}$$

amounts to (since $|y_i|^{1/2} = (-y_i)^{1/2} = x$)

$$-x^2 = y_{i+1} - hx$$

So

$$x^2 - hx + y_{i+1} = 0$$

So

$$x = \frac{h \pm \sqrt{h^2 - 4y_{i+1}}}{2}$$

Since $\sqrt{h^2 - 4y_{i+1}} > \sqrt{h^2} > h$, we have

$$x = \frac{h + \sqrt{h^2 - 4y_{i+1}}}{2}$$

so

$$y_i = -x^2 = -\left(\frac{h + \sqrt{h^2 - 4y_{i+1}}}{2}\right)^2$$

13c) Setting $y_{i+1} = -uh^2$, we have $(u>0)$ and

$$y_i = -\left(\frac{h + \sqrt{h^2 + 4uh^2}}{2}\right)^2$$

$$= -h^2 \frac{\left(1 + \sqrt{1+4u}\right)^2}{4}$$

$$= -h^2 \frac{1 + 2\sqrt{1+4u} + 1 + 4u}{4}$$

$$= -h^2 \frac{1 + 2u + \sqrt{1+4u}}{2}$$

4(a) $y_0 = -h^2$ (as per (3a))

(b) As per (4a), $y_1 = -h^2$ and so taking $u=1$ in (3c)

$$y_0 = -h^2 \frac{1+2\cdot 1+\sqrt{1+4\cdot 1}}{2}$$

$$= -h^2 \frac{3+\sqrt{5}}{2}$$

(c) We type "chaotic_sqrt(0,2,N,$y_0$);" for the pairs $(N, y_0)$ assigned, and note that MATLAB reports as follows:

| $(N, y_0)$ | $h = \frac{2}{N}$ | chaotic_sqrt(0,2,N,$y_0$); |
|---|---|---|
| $(2, -1)$ | $1$ | $y(2) = 0$ |
| $(4, -1/4)$ | $\frac{1}{2}$ | $y(2) = 0$ |
| $(16, -1/64)$ | $1/8$ | $y(2) = 0$ |
| $(64, -1/1024)$ | $1/32$ | $y(2) = 0$ |
| $(65, -\frac{4}{65^2})$ | $2/65$ | $y(2) = 0$ |
| $(63, \frac{-4}{63^2})$ | $2/63$ | $y(2) = 0.8041\ldots$ |
| $(99, \frac{-4}{99^2})$ | $2/99$ | $y(2) = 0$ |
| $(100, \frac{-4}{100^2})$ | $2/100$ | $y(2) = 0$ |
| $(101, \frac{-4}{101^2})$ | $2/101$ | $y(2) = 0.8702\ldots$ |

Hence the values where $N = 63, 101$ suffer from finite precision, since for $y_0 = -h^2$, the exact answer is $y_1 = y_2 = \cdots = 0$ by part (a)

4(d)

| $N$ | $y(2)$ |
|---|---|
| 7 | 0 |
| 8 | 0 |
| 9 | 0 |
| 10 | 0 |
| 100 | 0.8506... |
| 101 | 0 |
| $10^6$ | 0 |
| $10^6+1$ | 0.999992... |

Hence the values $N = 100,\ 10^6 + 1$ suffer from finite precision, by part (b).