CPSC 303, Homework Solutions

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(2a) $n=13$: largest entry of s-e_to_the_A
(in absolute value)
in homework2sol.txt is roughly $10^{-13}\cdot(-0.477)$

(2b) $n=13$, mysteriously, the analogue for

$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ has seemingly almost the same

largest entry in absolute value

(3a) Typing "chaotic_sqrt(-2,2,1000,-1);"

(assuming you have saved chaotic_sqrt.m

into the current working directory),

you get 1.00355...

And "chaotic_sqrt(-2,2,100000,-1);"

gives 1.00021...

[Note that this agrees with the solution:

$$y(t) = \left[\begin{cases} -\frac{1}{4} t^2 & \text{for } t \le 0 \\ \frac{1}{4} t^2 & \text{for } t \ge 0 \end{cases}\right]$$

3(b) chaotic_sqrt(0,2,N,0).

for any N returns 0

3(c) chaotic_sqrt(0,2,10000,1e-20);

gives 0.99844... , and

3(d) chaotic_sqrt(0,2,10000,1e-40);

gives 0.99817...

Note: if you keep at it, you might notice that

chaotic_sqrt(0,2,10000,1e-200);

returns 0.99765... and

chaotic_sqrt(0,2,10000,1e-400);

returns 0

This may seem strange...

(3e) Here there is no precise right or wrong answer...

The fact that

$$y' = |y|^{1/2} \qquad y(0) = 0$$

and

$$y' = |y|^{1/2} \qquad y(0) = 10^{-40}$$

have such different values at $t = 2$ (i.e., the computed value of $y(2)$) hints that $y' = |y|^{1/2}$ is very badly behaved near $y = 0$.

We have said (in class) that this is because $f(y) = |y|^{1/2}$ is not differentiable at $y = 0$.

Also, you may have noticed that when you solve

$$y' = f(y), \qquad y(t_0) = y_0,$$

then for any fixed t_1, the value of

$$y(t_1) \text{ which depends on } f, t_0, t_1, y_0$$

Depends continuously on y_0 — it certainly does if $f(y)$ is differentiable.

For example, the solution to

$$y' = Ay \qquad y(t_0) = y_0$$

is $y(t) = e^{A(t-t_0)} y_0$,

so

$$y(t_1) = e^{A(t_1 - t_0)} y_0$$

which is continuous in y_0 for

$\lfloor$ fixed A, t_1, t_0 $\rfloor$

(4a) If $z(t) = (\mathrm{Trans}_{T_2}\, y)(t)$

then $\boxed{z(t+T_1) = y(t) \quad (\forall t)}$ (*)

If $u(t) = (\mathrm{Trans}_{T_1}\, z)(t)$

(hence $u = \mathrm{Trans}_{T_1}(\mathrm{Trans}_{T_2}(y))$)

$\boxed{u(s+T_2) = z(s) \quad (\forall s)}$ (**)

Setting $s = t+T_1$, then (*) and (**) imply

$$u(t+T_1+T_2) = z(t+T_1) = y(t)$$

(for all t). Hence

$$u = \mathrm{Trans}_{T_1+T_2}(y).$$

(4b) Let $z = \text{Reverse}_T(y)$ and

$u = \text{Reverse}_T(z)$. Then

$$u = \text{Reverse}_T(z) = \text{Reverse}_T\left(\text{Reverse}_T(y)\right).$$

On the other hand

$$z(t) = y(2T - t) \qquad \forall t$$

$$u(s) = z(2T - s) \qquad \forall s$$

Setting $t = 2T - s$, we get

$$u(s) = z(2T - s) = y\left(2T - (2T - s)\right)$$

$$= y(s).$$

Hence $u = y$ (as functions), so

$$\text{Reverse}_T\left(\text{Reverse}_T(y)\right) = y$$

(4c) Similarly to (4b)

Let $z = \text{Reverse}_{T_2}(y)$ and

$u = \text{Reverse}_{T_1}(z)$. Then

$$u = \text{Reverse}_{T_1}(z) = \text{Reverse}_{T_1}\left(\text{Reverse}_{T_2}(y)\right).$$

On the other hand

$$z(t) = y(2T_2 - t) \quad \forall t$$

$$u(s) = z(2T_1 - s) \quad \forall s$$

Setting $t = 2T_1 - s$, we get

$$u(s) = z(2T_1 - s) = y\left(2T_2 - (2T_1 - s)\right)$$

$$= y(s + 2T_2 - 2T_1)$$

so

$$u(t - 2T_2 + 2T_1) = y(t)$$

So $\quad u = \text{Trans}_{(2T_1 - 2T_2)}(y)$.

(4d) Since $z(t) = y(t-T)$,

by the chain rule

$$\big(z(t)\big)' = \big(y'(t-T)\big)\,\big((t-T)'\big)$$

$$= y'(t-T)\cdot 1$$

$$= y'(t-T).$$

Hence

$$z'(t) = y'(t-T) = f\big(y(t-T)\big)$$

$$= f\big(z(t)\big).$$

(4e) If $z' = Az$, then

$$z(t) = e^{A(t-t_0)}\, z_0$$

Hence if $z = z(t)$ is positive any where, then $z_0 > 0$, and

$$z(t) = e^{A(t-t_0)}\, e^{\ln(z_0)}$$

$$= e^{A\left(t - t_0 - \frac{\ln(z_0)}{A}\right)}$$

provided that $A = 0$, Hence

$$z(t) = y\left(t - t_0 - \frac{\ln(z_0)}{A}\right)$$

So

$$z = \mathrm{Trans}_{\left(t_0 + \frac{\ln(z_0)}{A}\right)}(y).$$

(4f) Similarly, if $z(t) = y(2T - t)$, then, by the chain rule,

$$z'(t) = \big(y(2T-t)\big)' = y'(2T-t)\,(2T-t)'$$

$$= -y'(2T-t).$$

Hence

$$z'(t) = -y'(2T-t) = -f(y(2T-t))$$

$$= -f(z(t))$$

So z satisfies $z' = -f(z)$.

(4g) Continuing from (4f), since

$z'(t) = -y'(2T-t)$, the chain rule implies

$$z''(t) = (z'(t))' = -(y'(2T-t))'$$

$$= -y''(2T-t)(2T-t)'$$

$$= -y''(2T-t)(-1) = y''(2T-t)$$

Hence, if $y'' = f(y)$, then

$$z''(t) = y''(2T-t) = f(y(2T-t))$$

$$= f(z(t))$$

So z satisfies $z'' = f(z)$.